

ON THE DIRICHLET SEMIGROUP FOR ORNSTEIN – UHLENBECK OPERATORS IN SUBSETS OF HILBERT SPACES

GIUSEPPE DA PRATO AND ALESSANDRA LUNARDI

ABSTRACT. We consider a family of self-adjoint Ornstein–Uhlenbeck operators \mathcal{L}_α in an infinite dimensional Hilbert space H having the same gaussian invariant measure μ for all $\alpha \in [0, 1]$. We study the Dirichlet problem for the equation $\lambda\varphi - \mathcal{L}_\alpha\varphi = f$ in a closed set K , with $f \in L^2(K, \mu)$. We first prove that the variational solution, trivially provided by the Lax–Milgram theorem, can be represented, as expected, by means of the transition semigroup stopped to K . Then we address two problems: 1) the regularity of the solution φ (which is by definition in a Sobolev space $W_\alpha^{1,2}(K, \mu)$) of the Dirichlet problem; 2) the meaning of the Dirichlet boundary condition. Concerning regularity, we are able to prove interior $W_\alpha^{2,2}$ regularity results; concerning the boundary condition we consider both irregular and regular boundaries. In the first case we content to have a solution whose null extension outside K belongs to $W_\alpha^{1,2}(H, \mu)$. In the second case we exploit the Malliavin’s theory of surface integrals which is recalled in the Appendix of the paper, then we are able to give a meaning to the trace of φ at ∂K and to show that it vanishes, as it is natural.

1. INTRODUCTION AND SETTING OF THE PROBLEM

In this paper we present some results on second order elliptic and parabolic equations with Dirichlet boundary conditions in a closed set of a separable real Hilbert space H (norm $|\cdot|$, inner product $\langle \cdot, \cdot \rangle$).

A motivation for the study of Dirichlet problems in proper subsets of H is to provide a natural development of the potential theory in infinite dimensions started in [9]. Only a few results seem to be available in this field, see e.g. [5] and the references therein.

The finite dimensional theory in spaces of continuous functions is hardly extendable to the infinite dimensional setting. While in finite dimensions smooth boundaries consist only of regular points in the sense of Wiener, in infinite dimensions this is not true: for instance, certain hyperplanes and the boundary of the unit ball contain dense subsets of irregular points for suitable Ornstein–Uhlenbeck operators ([4]). This leads to the lack of regularity results up to the boundary.

Here we avoid a part of such difficulties working in suitable L^2 spaces.

To begin with, we consider a class of Ornstein–Uhlenbeck operators of the type

$$\mathcal{L}_\alpha\varphi(x) = \frac{1}{2} \operatorname{Tr} [Q^{1-\alpha} D^2\varphi(x)] - \frac{1}{2} \langle x, Q^{-\alpha} D\varphi(x) \rangle, \quad (1.1)$$

where $Q \in \mathcal{L}(H)$ is a symmetric positive operator with finite trace, and $0 \leq \alpha \leq 1$.

The most popular among such operators are \mathcal{L}_0 and \mathcal{L}_1 :

$$\mathcal{L}_0\varphi(x) = \frac{1}{2} \operatorname{Tr} [Q D^2\varphi(x)] - \frac{1}{2} \langle x, D\varphi(x) \rangle,$$

2000 *Mathematics Subject Classification.* Primary 35R15; Secondary 60HXX, 60H15.

Key words and phrases. Ornstein–Uhlenbeck operators, invariant measures, Dirichlet problems.

is the operator that arises in the Malliavin calculus, while

$$\mathcal{L}_1\varphi(x) = \frac{1}{2} \operatorname{Tr} [D^2\varphi(x)] - \frac{1}{2} \langle x, AD\varphi(x) \rangle,$$

(with $A = Q^{-1}$) is the generator of the Ornstein-Uhlenbeck semigroup with the best smoothing properties. See e.g. [5].

The operators \mathcal{L}_α exhibit an important common feature: the associated Ornstein-Uhlenbeck semigroups $T_\alpha(t)$ in $C_b(H)$ have the same invariant measure $\mu = \mathcal{N}_Q$, the Gaussian measure of mean 0 and covariance Q . In this paper we shall consider realizations of the operators \mathcal{L}_α in the space $L^2(K, \mu)$, where K is a closed set in H with non empty interior part \mathring{K} .

A unique weak solution to the Dirichlet problem

$$\begin{cases} \lambda\varphi(x) - \mathcal{L}_\alpha\varphi(x) = f(x), & \text{in } K, \\ \varphi(x) = 0, & \text{on } \partial K \end{cases} \quad (1.2)$$

with $\lambda > 0$ and $f \in L^2(K, \mu)$ is easily obtained via the Lax-Milgram Theorem, applied in a Hilbert space $\mathring{W}_\alpha^{1,2}(K, \mu)$ “naturally” associated to \mathcal{L}_α (see next section). This allows to define a dissipative self-adjoint operator M_α in $L^2(K, \mu)$ such that $\varphi = R(\lambda, M_\alpha)f$. As all dissipative self-adjoint operators in Hilbert spaces, M_α is the infinitesimal generator of an analytic contraction semigroup.

We give an explicit expression of the semigroup generated by M_α . Precisely, we identify it with the natural extension to $L^2(K, \mu)$ of the so-called *stopped semigroup* $T_\alpha^K(t)$. In analogy with the finite dimensional case (e.g., [8]), it is defined in $B_b(K)$ (the space of the bounded and Borel measurable functions defined in K) by

$$\begin{aligned} T_\alpha^K(t)\varphi(x) &= \mathbb{E}[\varphi(X_\alpha(t, x))\mathbb{1}_{\tau_x \geq t}] \\ &= \int_{\{\tau_x \geq t\}} \varphi(X_\alpha(t, x))d\mathbb{P}, \quad \forall x \in K, \end{aligned} \quad (1.3)$$

where τ_x is the entrance time in the complement of K ,

$$\tau_x := \inf\{t \geq 0 : X_\alpha(t, x) \in K^c\}, \quad \forall x \in K, \quad (1.4)$$

and $X_\alpha(t, x)$ is the solution to

$$dX_\alpha(t, x) = -\frac{1}{2}A^\alpha X_\alpha(t, x)dt + A^{(\alpha-1)/2}dW(t), \quad X(0, x) = x. \quad (1.5)$$

Here $W(t)$ is a standard cylindrical Wiener process in H , defined in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

The definition of $T_\alpha^K(t)$ is similar to the one in [15], where the exit time from \mathring{K} , $\tilde{\tau}_x := \inf\{t \geq 0 : X_\alpha(t, x) \in \mathring{K}^c\}$ was used instead of our τ_x . In finite dimensions, if K is the closure of a bounded open set with smooth boundary the two definitions are equivalent, and $T_\alpha^K(t)$ is the semigroup associated to the realization of \mathcal{L}_α with Dirichlet boundary condition ([8, §6.5]). Therefore, a lot of regularity results, both interior and up to the boundary, are well known. In infinite dimensions, interior regularity results were given in [15] for $\alpha > 0$. We do not know regularity results up to the boundary, even in the case of very smooth bounded sets such as balls.

Here we prove that μ is a sub-invariant measure for $T_\alpha^K(t)$. Therefore, $T_\alpha^K(t)$ has a natural extension (still called $T_\alpha^K(t)$) to a contraction semigroup in $L^2(K, \mu)$. The

domain of its generator L_α^K consists of the range of the resolvent operator,

$$R(\lambda, L_\alpha^K)f = \int_0^\infty e^{-\lambda t} T_\alpha^K(t)f dt, \quad f \in L^2(K, \mu), \quad (1.6)$$

which is well defined for $\lambda > 0$ since $T_\alpha^K(t)$ is a contraction semigroup. We prove that for each $\lambda > 0$ and $f \in L^2(K, \mu)$, the function $\varphi := R(\lambda, L_\alpha^K)f$ belongs to the above mentioned space $\dot{W}_\alpha^{1,2}(K, \mu)$, and satisfies the weak formulation of (1.2). Therefore, $L_\alpha^K = M_\alpha$.

Our main tool in the proof is the approximating Feynman–Kac semigroup

$$P_\alpha^\varepsilon(t)\varphi(x) = \mathbb{E} \left[\varphi(X_\alpha(t, x)) e^{-\frac{1}{\varepsilon} \int_0^t V(X_\alpha(s, x)) ds} \right], \quad (1.7)$$

where V is a (fixed) bounded continuous function that vanishes in K and has positive values in K^c . Its infinitesimal generator in $L^2(H, \mu)$ is the operator $M_\alpha^\varepsilon : D(M_\alpha^\varepsilon) = D(L_\alpha) \mapsto L^2(H, \mu)$, $M_\alpha^\varepsilon \varphi = L_\alpha \varphi_\varepsilon - \frac{1}{\varepsilon} V \varphi$, and we prove that for each $\varphi \in L^2(K, \mu)$, $t > 0$, $\lambda > 0$ we have

$$T_\alpha^K(t)\varphi = \lim_{\varepsilon \rightarrow 0} (P_\alpha^\varepsilon(t)\tilde{\varphi})|_K, \quad R(\lambda, L_\alpha^K)\varphi = \lim_{\varepsilon \rightarrow 0} (R(\lambda, M_\alpha^\varepsilon)\tilde{\varphi})|_K$$

in $L^2(K, \mu)$, where $\tilde{\varphi}$ is the null extension of φ to the whole H .

Problem (1.2) is of interest for $\lambda = 0$ too. Using the fact that $D(L_\alpha^K)$ is compactly embedded in $L^2(K, \mu)$, in Sect. 3.3 we prove that $0 \in \rho(L_\alpha^K)$ and that a Poincaré estimate holds in $\dot{W}_\alpha^{1,2}(K, \mu)$, for $\alpha \in (0, 1]$. Therefore, the supremum of $\sigma(L_\alpha^K)$ is negative.

These results are proved without additional assumptions on K . In particular, we do not require that K is bounded, or that its boundary is smooth.

If the boundary of K is suitably smooth, it is possible to define surface integrals and traces at the boundary of functions in the Sobolev spaces $W_\alpha^{1,2}(K, \mu)$. Then we prove that the traces of the functions in $\dot{W}_\alpha^{1,2}(K, \mu)$ vanish. Therefore, the Dirichlet boundary condition in (1.2) is satisfied in the sense of the trace, and $T_\alpha^K(t)\varphi$ has null trace at the boundary for every $t > 0$ and $\varphi \in L^2(K, \mu)$.

Surface integrals for gaussian measures in Hilbert spaces are not a straightforward extension of the finite dimensional theory. To our knowledge the best reference is [2, §6.10], where the Malliavin theory is presented. It deals with level surfaces of smooth functions g in a more general context than ours, since Souslin spaces X are considered instead of Hilbert spaces. A part of the theory may be simplified in our Hilbert setting, and moreover some of the smoothness assumptions on g can be weakened. Therefore, we end the paper with an appendix describing surface measures for level surfaces of suitably regular functions $g : H \mapsto \mathbb{R}$.

Several related important problems remain open, even for bounded K with smooth boundary. Among them:

- (a) While in finite dimensions $\varphi = R(\lambda, L_\alpha^K)f$ is a strong solution to (1.2) and it belongs to $W^{2,2}(K, \mu)$ under reasonable assumptions on the boundary ∂K ([13]), in infinite dimensions we do not know whether φ possesses second order derivatives in $L^2(K, \mu)$, even if K is the closed unit ball. In fact, even in the case $\alpha = 1$, the estimates found in [4, 15] are very bad both near the boundary and near $t = 0$, and it is not clear how to use them to get informations on the resolvent.

- (b) We do not know whether $T_\alpha^K(t)$ is strong Feller in K (i.e., it maps $B_b(K)$, the space of the bounded Borel functions in K , to $C_b(K)$). This problem is open even for $K = \{x \in H : |x| \leq 1\}$.
- (c) In finite dimensions, if ∂K is regular enough there are several characterizations of the space $\mathring{W}_\alpha^{1,2}(K, \mu)$, that coincides with $\mathring{W}_1^{1,2}(K, \mu)$ for every $\alpha \in [0, 1]$. The most obvious is the following: since μ is locally equivalent to the Lebesgue measure, $\mathring{W}_1^{1,2}(K, \mu)$ coincides with the space of the functions $f \in W_1^{1,2}(K, \mu)$ whose trace at the boundary vanishes. We do not know whether a similar characterization holds in infinite dimensions.

Referring to problem (a), in the recent paper [1] a self-adjoint realization L of \mathcal{L}_1 in $L^2(K, \mu)$ with Neumann boundary condition has been studied, in the case that K is a convex set with regular boundary. By means of a different (and better) approximation procedure, it has been proved that the resolvent $R(\lambda, L)$ maps $L^2(K, \mu)$ into $W_1^{2,2}(K, \mu)$.

Here we prove interior $W_\alpha^{2,2}$ regularity, for those α such that $\text{Tr}[Q^{1-\alpha}] < \infty$. In this case we show that for every ball $B \subset K$ with positive distance from ∂K and for every $\varphi \in D(L_\alpha^K)$, the restriction $\varphi|_B$ belongs to $W_\alpha^{2,2}(B, \mu)$.

2. NOTATION AND PRELIMINARIES

We denote by $\langle \cdot, \cdot \rangle$ and by $|\cdot|$ the scalar product and the norm in H . $\mathcal{L}(H)$ is the space of the linear bounded operators in H .

Let Q be a symmetric (strictly) positive operator in $\mathcal{L}(H)$ with finite trace, and let $A := Q^{-1}$. Accordingly, let $\{e_k\}$ be an orthonormal basis in H consisting of eigenfunctions of Q , i.e.

$$Qe_k = \lambda_k e_k, \quad Ae_k = \frac{1}{\lambda_k} e_k, \quad \forall k \in \mathbb{N}.$$

We denote by D_k the derivative in the direction of e_k and by D the gradient of any differentiable function. Moreover we set $x_k = \langle x, e_k \rangle$ for all $x \in H$, $k \in \mathbb{N}$.

Throughout the paper we consider the σ -algebra $\mathcal{B}(H)$ of the Borel subsets of H and the Gaussian measure with center 0 and covariance Q in $\mathcal{B}(H)$, denoting it by μ .

An orthonormal basis of $L^2(H, \mu)$ consists of the Hermite polynomials. More precisely, for each $n \in \mathbb{N} \cup \{0\}$ let

$$H_n(\xi) := (-1)^n n!^{-1/2} e^{\xi^2/2} D^n(e^{-\xi^2/2}), \quad \xi \in \mathbb{R},$$

be the usual normalized n -th Hermite polynomial. We denote by Γ the set of all $\gamma : \mathbb{N} \mapsto \mathbb{N} \cup \{0\}$ such that $\sum_{k=1}^\infty \gamma(k) < \infty$. For each $\gamma \in \Gamma$ let

$$H_\gamma(x) := \prod_{k=1}^\infty H_{\gamma(k)}\left(\frac{x_k}{\sqrt{\lambda_k}}\right), \quad x \in H,$$

be the corresponding Hermite polynomial in H . Then, the linear span \mathcal{H} of all the Hermite polynomials H_γ is dense in $L^2(H, \mu)$, and the linear span Λ_0 of the functions $H_\gamma \otimes e_h$, with $\gamma \in \Gamma$ and $h \in \mathbb{N}$, is dense in the space $L^2(H, \mu; H)$ of all the (equivalence classes of) measurable functions $F : H \mapsto H$ such that $\int_H |F(x)|^2 \mu(dx) < \infty$.

Other important dense subspaces of $L^2(H, \mu)$ are the spaces $\mathcal{E}_\alpha(H)$, the linear spans of the real and imaginary parts of the functions $x \mapsto e^{i\langle x, h \rangle}$, with $h \in D(A^\alpha)$, $0 \leq \alpha \leq 1$.

2.1. Sobolev spaces over H . We have the following integration formula,

$$\int_H D_k \varphi d\mu = \frac{1}{\lambda_k} \int_H x_k \varphi d\mu, \quad \varphi \in \mathcal{E}_\alpha(H), \quad k \in \mathbb{N}. \quad (2.1)$$

It may be extended to

$$\int_H \langle D\varphi, G \rangle d\mu + \int_H \varphi \operatorname{div} G d\mu = \int_H \varphi \langle x, AG(x) \rangle d\mu, \quad \varphi \in C_b^1(H), \quad G \in \Lambda_0, \quad (2.2)$$

where $\operatorname{div} G(x) = \sum_{k=1}^\infty \langle DG(x), e_k \rangle$. The linear operator $Q^{(1-\alpha)/2}D$ is well defined from $\mathcal{E}_\alpha(H) \subset L^2(H, \mu)$ to $L^2(H, \mu; H)$, by

$$Q^{(1-\alpha)/2}D\varphi = \sum_{k=1}^\infty \lambda_k^{(1-\alpha)/2} D_k \varphi e_k.$$

Using formula (2.2) it is easy to see that $Q^{(1-\alpha)/2}D$ is closable. We still denote by $Q^{(1-\alpha)/2}D$ its closure, and by $W_\alpha^{1,2}(H, \mu)$ the domain of the closure. (Note that for $\alpha = 0$, $Q^{1/2}D$ is nothing but the Malliavin derivative). $W_\alpha^{1,2}(H, \mu)$ is endowed with the inner product

$$\begin{aligned} \langle \varphi, \psi \rangle_{W_\alpha^{1,2}(H, \mu)} &= \int_H \varphi \psi d\mu + \int_H \langle Q^{(1-\alpha)/2}D\varphi, Q^{(1-\alpha)/2}D\psi \rangle d\mu \\ &= \int_H \varphi \psi d\mu + \sum_{k=1}^\infty \int_H \lambda_k^{1-\alpha} D_k \varphi D_k \psi d\mu. \end{aligned} \quad (2.3)$$

So, $W_\alpha^{1,2}(H, \mu)$ is the completion of $\mathcal{E}_\alpha(H)$ in the norm associated to the scalar product (2.3). It is also possible to characterize it through the Hermite polynomials. We have $\varphi \in W_\alpha^{1,2}(H, \mu)$ iff

$$\sum_{\gamma \in \Gamma} \sum_{h=1}^\infty \gamma_h \lambda_h^{-\alpha} \varphi_\gamma^2 < \infty$$

in which case the above sum is equal to $\int_H |Q^{(1-\alpha)/2}D\varphi|^2 d\mu$. Indeed, the proof in [5, Sect. 9.2.3] for $\alpha = 1$ works as well for any $\alpha \in [0, 1]$.

From this characterization it is clear that $W_\alpha^{1,2}(H, \mu) \subset W_0^{1,2}(H, \mu)$ for every $\alpha \in (0, 1]$, with continuous embedding.

Similarly, $W_\alpha^{2,2}(H, \mu)$ is the completion of $\mathcal{E}_\alpha(H)$ in the norm associated to the scalar product

$$\begin{aligned} \langle \varphi, \psi \rangle_{W_\alpha^{2,2}(H, \mu)} &= \langle \varphi, \psi \rangle_{W_\alpha^{1,2}(H, \mu)} + \int_H \operatorname{Tr} [Q^{2-2\alpha} D^2 \varphi D^2 \psi] d\mu \\ &= \langle \varphi, \psi \rangle_{W_\alpha^{1,2}(H, \mu)} + \sum_{h,k=1}^\infty \int_H \lambda_h^{1-\alpha} \lambda_k^{1-\alpha} D_{h,k} \varphi D_{h,k} \psi d\mu. \end{aligned}$$

Next lemma is a consequence of [2, Lemma 5.1.12] or [5, Lemma 9.2.7].

Lemma 2.1. *There is $C > 0$ such that*

$$\int_H |x|^2 \varphi(x)^2 d\mu \leq C \|\varphi\|_{W_0^{1,2}(H, \mu)}^2, \quad \varphi \in W_0^{1,2}(H, \mu).$$

Lemma 2.1, together with (2.1), yields the integration by parts formula in $W_0^{1,2}(H, \mu)$ (and hence, in all spaces $W_\alpha^{1,2}(H, \mu)$),

$$\int_H D_k \varphi \psi d\mu = - \int_H \varphi D_k \psi d\mu + \frac{1}{\lambda_k} \int_H x_k \varphi \psi d\mu, \quad \varphi, \psi \in W_0^{1,2}(H, \mu), \quad k \in \mathbb{N}. \quad (2.4)$$

For $0 \leq \alpha \leq 1$ let $T_\alpha(t)$ be the Ornstein-Uhlenbeck semigroup

$$T_\alpha(t)\varphi(x) := \int_H \varphi(y) \mathcal{N}_{e^{-tA^\alpha}/2x, Q_t}(dy), \quad t > 0, \quad (2.5)$$

with

$$Q_t := \int_0^t e^{-sA^\alpha} Q^{1-\alpha} ds = Q(I - e^{-tA^\alpha}).$$

$T_\alpha(t)$ is a Markov semigroup in $C_b(H)$, whose unique invariant measure is μ . Its extension to $L^2(H, \mu)$ is a strongly continuous contraction semigroup, still denoted by $T_\alpha(t)$, whose infinitesimal generator L_α is the closure of $\mathcal{L}_\alpha : \mathcal{E}_\alpha(H) \mapsto L^2(H, \mu)$.

The domain of L_α is continuously embedded in $W_\alpha^{2,2}(H, \mu)$. Moreover, for any $\varphi, \psi \in D(L_\alpha)$ we have

$$\int_H L_\alpha \varphi \psi d\mu = -\frac{1}{2} \int_H \langle Q^{(1-\alpha)/2} D\varphi, Q^{(1-\alpha)/2} D\psi \rangle d\mu. \quad (2.6)$$

We refer to [5, Ch. 9, 10] for the proofs of the above statements, and we add further properties of the spaces $W_\alpha^{1,2}(H, \mu)$ that will be used later. For each $\varphi \in L^1(H, \mu)$ we denote by $\bar{\varphi}$ the mean value of φ ,

$$\bar{\varphi} := \int_H \varphi d\mu.$$

Proposition 2.2. *Let $0 \leq \alpha \leq 1$. Then*

(a) *A Poincaré estimate holds in $W_\alpha^{1,2}(H, \mu)$, and precisely*

$$\int_H (\varphi - \bar{\varphi})^2 d\mu \leq \lambda_1^\alpha \int_H |Q^{(1-\alpha)/2} D\varphi|^2 d\mu, \quad (2.7)$$

where λ_1 is the maximum eigenvalue of Q .

(b) *The space $W_\alpha^{1,2}(H, \mu)$ is compactly embedded in $L^2(H, \mu)$ for $\alpha > 0$.*

Proof. A proof of statement (a) that follows the approach of Deuschel and Strook [6] is in [5, Ch.10] for $\alpha = 1$. The same procedure works for $\alpha \in [0, 1)$, since the key points of the proof still hold. Precisely, we have

(i) $|Q^{(1-\alpha)/2} D T^\alpha(t) \varphi|^2 \leq e^{-t/\lambda_1^\alpha} T^\alpha(t)(|Q^{(1-\alpha)/2} D\varphi|^2)$, $\varphi \in C_b^1(H)$, $t > 0$;

(ii) $\int_H \varphi L_\alpha \varphi d\mu = -\frac{1}{2} \int_H |Q^{(1-\alpha)/2} D\varphi|^2 d\mu$, $\varphi \in D(L_\alpha)$;

(iii) $\lim_{t \rightarrow \infty} T^\alpha(t) \varphi(x) = \bar{\varphi}$, $\varphi \in \mathcal{E}_\alpha(H)$, $x \in H$.

Once (i), (ii), (iii) are satisfied one can follow the proof of [5, Prop. 10.5.2] step by step. (ii) and (iii) follow from [5, Prop. 10.2.3, Prop. 10.1.1]. To check that (i) holds is easy and it is left to the reader.

Statement (b) should be well known, however we give here a simple proof following [3, Thm. 10.16] that concerns the case $\alpha = 1$. We write every element φ of $L^2(H, \mu)$ as $\varphi = \sum_{\gamma \in \Gamma} \varphi_\gamma H_\gamma$, with $\varphi_\gamma = \langle \varphi, H_\gamma \rangle$. We already remarked that $\varphi \in W_\alpha^{1,2}(H, \mu)$ iff

$$\sum_{\gamma \in \Gamma} \sum_{h=1}^{\infty} \gamma_h \lambda_h^{-\alpha} \varphi_\gamma^2 < \infty.$$

If a sequence $(\varphi^{(n)})$ is bounded in $W_\alpha^{1,2}(H, \mu)$, say $\|\varphi^{(n)}\|_{W_\alpha^{1,2}(H, \mu)} \leq K$ for each $n \in \mathbb{N}$, a subsequence $(\varphi^{(n_k)})$ converges weakly in $W_\alpha^{1,2}(H, \mu)$ to a limit φ , that still satisfies $\|\varphi\|_{W_\alpha^{1,2}(H, \mu)} \leq K$. We shall show that $\lim_{k \rightarrow \infty} \|\varphi^{(n_k)} - \varphi\|_{L^2(H, \mu)} = 0$.

For each $N \in \mathbb{N}$, let $\Gamma_N = \{\gamma \in \Gamma : \sum_{h=1}^\infty \gamma_h \lambda_h^{-\alpha} < \infty\}$. Then

$$\begin{aligned} \int_H (\varphi^{(n_k)} - \varphi)^2 d\mu &= \sum_{\gamma \in \Gamma_N} (\varphi_\gamma^{(n_k)} - \varphi_\gamma)^2 + \sum_{\gamma \in \Gamma_N^c} (\varphi_\gamma^{(n_k)} - \varphi_\gamma)^2 \\ &\leq \sum_{\gamma \in \Gamma_N} (\varphi_\gamma^{(n_k)} - \varphi_\gamma)^2 + \frac{1}{N} \sum_{\gamma \in \Gamma} \sum_{h=1}^\infty \gamma_h \lambda_h^{-\alpha} (\varphi_\gamma^{(n_k)} - \varphi_\gamma)^2 \\ &\leq \sum_{\gamma \in \Gamma_N} (\varphi_\gamma^{(n_k)} - \varphi_\gamma)^2 + \frac{(2K)^2}{N}. \end{aligned}$$

For $\varepsilon > 0$ fix $N \in \mathbb{N}$ such that $4K^2/N \leq \varepsilon$. Since $\alpha > 0$, then $\lim_{h \rightarrow \infty} \lambda_h^{-\alpha} = +\infty$, so that the set Γ_N has a finite number of elements. Since $\varphi^{(n_k)}$ converges weakly to φ in $W_\alpha^{1,2}(H, \mu)$, it converges weakly to φ in $L^2(H, \mu)$; in particular $\lim_{h \rightarrow \infty} \varphi_\gamma^{(n_k)} = \varphi_\gamma$ for each $\gamma \in \Gamma_N$. Therefore, for k large enough we have $\sum_{\gamma \in \Gamma_N} (\varphi_\gamma^{(n_k)} - \varphi_\gamma)^2 \leq \varepsilon$, and the statement follows. \square

2.2. Sobolev spaces over K . Throughout the paper we assume that $K \subset H$ is a closed set with positive measure. To avoid trivialities, we assume that also K^c has positive measure.

To treat the Dirichlet problem (1.2) we introduce Sobolev spaces over K . We denote by $W_\alpha^{1,2}(K, \mu)$ the space of the functions $u : K \mapsto \mathbb{R}$ that have an extension belonging to $W_\alpha^{1,2}(H, \mu)$, endowed with the standard inf norm. Moreover we denote by $\dot{W}_\alpha^{1,2}(K, \mu)$ the subspace of $W_\alpha^{1,2}(K, \mu)$ consisting of the functions $u : K \mapsto \mathbb{R}$ whose null extension to the whole H belongs to the Sobolev space $W_\alpha^{1,2}(H, \mu)$. Therefore,

$$\|u\|_{W_\alpha^{1,2}(K, \mu)}^2 = \int_K u^2 d\mu + \int_K |Q^{(1-\alpha)/2} Du|^2 d\mu, \quad u \in \dot{W}_\alpha^{1,2}(K, \mu),$$

so that the $W_\alpha^{1,2}(K, \mu)$ -norm in $\dot{W}_\alpha^{1,2}(K, \mu)$ is associated to the inner product

$$\langle u, v \rangle_{W_\alpha^{1,2}(K, \mu)} = \int_K u v d\mu + \int_K \langle Q^{(1-\alpha)/2} Du, Q^{(1-\alpha)/2} Dv \rangle d\mu. \quad (2.8)$$

From the results of the next section it will be clear that such a space is not trivial, since it coincides with the domain of $(I - L_\alpha^K)^{1/2}$. Moreover, since $W_\alpha^{1,2}(H, \mu)$ is continuously embedded in $W_0^{1,2}(H, \mu)$, then $\dot{W}_\alpha^{1,2}(K, \mu)$ is continuously embedded in $\dot{W}_0^{1,2}(K, \mu)$, for every $\alpha \in (0, 1]$.

2.3. The weak solution to (1.2). The quadratic form \mathcal{Q}_α associated to \mathcal{L}_α ,

$$\mathcal{Q}_\alpha(u, v) := \frac{1}{2} \int_K \langle Q^{(1-\alpha)/2} Du, Q^{(1-\alpha)/2} Dv \rangle d\mu, \quad u, v \in \dot{W}_\alpha^{1,2}(K, \mu), \quad (2.9)$$

is continuous, nonnegative, and symmetric. Therefore, for every $\lambda > 0$ and $f \in L^2(K, \mu)$ there exists a unique $\varphi \in \dot{W}_\alpha^{1,2}(K, \mu)$ such that

$$\lambda \int_K \varphi v d\mu + \frac{1}{2} \int_K \langle Q^{(1-\alpha)/2} D\varphi, Q^{(1-\alpha)/2} Dv \rangle d\mu = \int_K f v d\mu, \quad \forall v \in \dot{W}_\alpha^{1,2}(K, \mu). \quad (2.10)$$

The function φ may be considered a weak solution to (1.2). Moreover, there exists a dissipative self-adjoint operator M_α in $L^2(K, \mu)$ such that $\varphi = R(\lambda, M_\alpha)f$. Like all dissipative self-adjoint operators in Hilbert spaces, M_α is the infinitesimal generator of an analytic contraction semigroup, and several properties of M_α follow. See e.g. [11, Ch. 6].

3. THE DIRICHLET SEMIGROUP

In this section we give an explicit representation formula for the semigroup generated by the operator M_α defined in section 2.3, through the approximation procedure described in the introduction. Moreover we show some properties of the semigroup and of its generator.

3.1. The approximating semigroups. We fix once and for all a function $V \in C_b(H)$ such that

$$V(x) = 0, \quad x \in K, \quad V(x) > 0, \quad x \in K^c.$$

For $\varepsilon > 0$ let $P_\alpha^\varepsilon(t)$ be defined by (1.7).

Proposition 3.1. *For any $\varphi \in C_b(H)$ we have*

$$\int_H (P_\alpha^\varepsilon(t)\varphi(x))^2 \mu(dx) \leq \int_H \varphi^2(x) \mu(dx). \quad (3.1)$$

Consequently, $P_\alpha^\varepsilon(t)$ is uniquely extendable to a C_0 -semigroup in $L^2(H, \mu)$ which we shall denote by the same symbol.

Proof. We have in fact, by the Hölder inequality

$$(P_\alpha^\varepsilon(t)\varphi(x))^2 \leq \mathbb{E} \left(\varphi^2(X_\alpha(t, x)) e^{-\frac{2}{\varepsilon} \int_0^t V(X_\alpha(s, x)) ds} \right) \leq T_\alpha(t)(\varphi^2)(x),$$

where $T_\alpha(t)$ is the Ornstein-Uhlenbeck semigroup defined in (2.5). Since μ is invariant for $T_\alpha(t)$, then

$$\int_H (P_\alpha^\varepsilon(t)\varphi(x))^2 \mu(dx) \leq \int_H T_\alpha(t)(\varphi^2)(x) \mu(dx) = \int_H \varphi^2(x) \mu(dx). \quad (3.2)$$

□

We denote by M_α^ε the infinitesimal generator of $P_\alpha^\varepsilon(t)$ in $L^2(H, \mu)$ and we want to show that $M_\alpha^\varepsilon = L_\alpha - \frac{1}{\varepsilon} V$. To this aim, for $\lambda > 0$ and $f \in L^2(H, \mu)$ we consider the resolvent equation

$$\lambda \varphi_\varepsilon - L_\alpha \varphi_\varepsilon + \frac{1}{\varepsilon} V \varphi_\varepsilon = f. \quad (3.3)$$

Proposition 3.2. *Let $\lambda > 0$, $\varepsilon > 0$, and $f \in L^2(H, \mu)$. Then equation (3.3) has a unique solution $\varphi_\varepsilon \in D(L_\alpha)$, and the following estimates hold.*

$$\int_H \varphi_\varepsilon^2 d\mu \leq \frac{1}{\lambda^2} \int_H f^2 d\mu, \quad (3.4)$$

$$\int_H |Q^{(1-\alpha)/2} D\varphi_\varepsilon|^2 d\mu \leq \frac{2}{\lambda} \int_H f^2 d\mu, \quad (3.5)$$

$$\int_{K^c} V \varphi_\varepsilon^2 d\mu \leq \frac{\varepsilon}{\lambda} \int_H f^2 d\mu. \quad (3.6)$$

Proof. Fix $\lambda > 0$ and $\varepsilon > 0$. Since L_α is maximal dissipative and $\varphi \rightarrow \frac{1}{\varepsilon} V\varphi$ is bounded and monotone increasing in $L^2(H, \mu)$, it follows by standard arguments that the operator

$$D(L_\alpha) \mapsto L^2(H, \mu), \quad \varphi \mapsto L_\alpha \varphi - \frac{1}{\varepsilon} V\varphi,$$

is maximal dissipative. So, equation (3.3) has a unique solution $\varphi_\varepsilon \in D(L_\alpha)$, that satisfies (3.4).

Multiplying both sides of (3.3) by φ_ε , integrating over H and taking into account (2.6) yields

$$\lambda \int_H \varphi_\varepsilon^2 d\mu + \frac{1}{2} \int_H |Q^{(1-\alpha)/2} D\varphi_\varepsilon|^2 d\mu + \frac{1}{\varepsilon} \int_{K^c} V\varphi_\varepsilon^2 d\mu = \int_H f\varphi_\varepsilon d\mu. \quad (3.7)$$

The inequality $\lambda \int_H |\varphi_\varepsilon|^2 d\mu \leq \int_H f\varphi_\varepsilon d\mu$ yields again (3.4). The inequality

$$\frac{1}{2} \int_H |Q^{(1-\alpha)/2} D\varphi_\varepsilon|^2 d\mu \leq \int_H f\varphi_\varepsilon d\mu$$

implies (3.5), using the Hölder inequality in the right-hand side and then (3.4). The inequality

$$\frac{1}{\varepsilon} \int_{K^c} V\varphi_\varepsilon^2 d\mu \leq \int_H f\varphi_\varepsilon d\mu$$

implies (3.6), using again the Hölder inequality in the right-hand side and then (3.4). \square

Proposition 3.3. *Let M_α^ε be the infinitesimal generator of $P_\alpha^\varepsilon(t)$. Then $D(M_\alpha^\varepsilon) = D(L_\alpha)$ and*

$$M_\alpha^\varepsilon \varphi = L_\alpha \varphi - \frac{1}{\varepsilon} V\varphi, \quad \forall \varphi \in D(L_\alpha). \quad (3.8)$$

Proof. Let us show that $D(L_\alpha) \subset D(M_\alpha^\varepsilon)$, and that (3.8) holds.

First, let $\varphi \in D(L_\alpha) \cap C_b(H)$. For $x \in H$, $h > 0$ we have

$$P_h^\varepsilon \varphi(x) - \varphi(x) = T_\alpha(h)\varphi(x) - \varphi(x) + \mathbb{E} \left[\left(e^{-\frac{1}{\varepsilon} \int_0^h V(X_\alpha(r, x)) dr} - 1 \right) \varphi(X_\alpha(h, x)) \right]. \quad (3.9)$$

We recall that, since A^α is self-adjoint, $X_\alpha(\cdot, x)$ possesses a.s. continuous paths ([12, 16]). Therefore the functions $r \mapsto \varphi(X_\alpha(r, x))$ and $r \mapsto V(X_\alpha(r, x))$ are continuous a.s. Dividing both sides of (3.9) by h and letting $h \rightarrow 0$, we obtain $\lim_{h \rightarrow 0} (P_h^\varepsilon \varphi - \varphi)/h = L_\alpha \varphi - V\varphi/\varepsilon$ pointwise and (by dominated convergence) in $L^2(H, \mu)$, so that $\varphi \in D(M_\alpha^\varepsilon)$ and (3.8) holds.

Let now $\varphi \in D(L_\alpha)$, and let (φ_n) be a sequence of functions in $\mathcal{E}_\alpha(H)$ that converges to φ in $D(L_\alpha)$. Then, $\varphi_n \rightarrow \varphi$ in $L^2(H, \mu)$, so that $\frac{1}{\varepsilon} V\varphi_n \rightarrow \frac{1}{\varepsilon} V\varphi$ in $L^2(H, \mu)$, moreover $L_\alpha \varphi_n \rightarrow L_\alpha \varphi$ in $L^2(H, \mu)$. It follows that $M_\alpha^\varepsilon \varphi_n \rightarrow M_\alpha^\varepsilon \varphi$ in $L^2(H, \mu)$, and since M_α^ε is closed, then $\varphi \in D(M_\alpha^\varepsilon)$ and (3.8) holds.

The other inclusion $D(M_\alpha^\varepsilon) \subset D(L_\alpha)$ is immediate. Indeed, for any $\varphi \in D(M_\alpha^\varepsilon)$ set $f = \lambda\varphi - M_\alpha^\varepsilon \varphi$, and let φ_ε be the solution of (3.3). Then $\varphi_\varepsilon \in D(L_\alpha) \subset D(M_\alpha^\varepsilon)$, so that $(\lambda - M_\alpha^\varepsilon)^{-1} f = \varphi_\varepsilon = \varphi$ which implies that $\varphi \in D(L_\alpha)$. \square

Remark 3.4. From the very beginning, one would be tempted to replace the continuous function V by $\mathbb{1}_{K^c}$ in the definition of M_ε . But with this choice the proof of Proposition 3.3 does not work. Indeed, it is not obvious that $(P_h^\varepsilon \varphi - \varphi)/h$ converges as $h \rightarrow 0$ for any $\varphi \in C_b(H) \cap D(L_\alpha)$, if $x \in \partial K$, because the function $r \mapsto \mathbb{1}_{K^c}(X_\alpha(r, x))$ could be discontinuous at $r = 0$. If $\mu(\partial K) = 0$ this difficulty is not relevant, since we are interested in L^2 convergence rather than in pointwise convergence. However, we prefer to make no further assumptions on ∂K in this first part of the paper.

3.2. Identification of $T_\alpha^K(t)$. Let $T_\alpha^K(t)$, $P_\alpha^\varepsilon(t)$ be defined by (1.3), (1.7) respectively.

Proposition 3.5. *For any $\varphi \in B_b(H)$, $t > 0$, and for any $x \in K$ we have*

$$\lim_{\varepsilon \rightarrow 0} P_\alpha^\varepsilon(t)\varphi(x) = T_\alpha^K(t)\varphi|_K(x). \quad (3.10)$$

Moreover $T_\alpha^K(t)$ is a semigroup of linear bounded operators in $B_b(K)$.

Proof. Let $t > 0$, $x \in K$. Then

$$\{\tau_x^K \geq t\} = \{\omega \in \Omega : X_\alpha(s, x) \in K, \forall s \in [0, t]\}$$

and

$$\{\tau_x^K < t\} = \{\omega \in \Omega : \exists s_0 \in (0, t) : X_\alpha(s_0, x) \in K^c\}$$

Then we have

$$P_\alpha^\varepsilon(t)\varphi(x) = \int_{\{\tau_x^K \geq t\}} \varphi(X_\alpha(t, x)) d\mathbb{P} + \int_{\{\tau_x^K < t\}} \varphi(X_\alpha(t, x)) e^{-\frac{1}{\varepsilon} \int_0^t V(X_\alpha(s, x)) ds} d\mathbb{P}$$

In view of the dominated convergence theorem, to prove the statement it is enough to show that

$$\lim_{\varepsilon \rightarrow 0} e^{-\frac{1}{\varepsilon} \int_0^t V(X_\alpha(s, x)) ds} = 0, \quad (3.11)$$

for a.a. ω such that $\tau_x^K(\omega) < t$.

We already mentioned that $X_\alpha(\cdot, x)$ possesses a.s. continuous paths. Let $\omega \in \Omega$ be such that $X_\alpha(\cdot, x)(\omega)$ is continuous. If $\tau_x^K(\omega) < t$, there exist $s_0 < t$, $\delta > 0$ (depending on ω) such that

$$X_\alpha(s, x) \in K^c, \quad \forall s \in [s_0 - \delta, s_0 + \delta].$$

Since V is continuous and it has positive values in K^c , then

$$c := \inf\{V(X(s, x)) : s \in [s_0 - \delta, s_0 + \delta]\} > 0.$$

It follows that

$$e^{-\frac{1}{\varepsilon} \int_0^t V(X_\alpha(s, x)) ds} \leq e^{-\frac{2c}{\varepsilon} \delta} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

So, (3.11) holds. The last statement is straightforward. \square

In the next proposition we show that μ is sub-invariant for $T_\alpha^K(t)$. We use the following notation. For each $\varphi \in B_b(K)$ we set

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & \text{if } x \in K, \\ 0, & \text{if } x \notin K. \end{cases}$$

Proposition 3.6. *For any $\varphi \in B_b(K)$, $t > 0$, we have*

$$\int_K (T_\alpha^K(t)\varphi(x))^2 \mu(dx) \leq \int_K \varphi^2(x) \mu(dx). \quad (3.12)$$

Consequently, $T_\alpha^K(t)$ can be uniquely extended to a C_0 semigroup of contractions in $L^2(K, \mu)$.

Proof. By the Hölder inequality we have for all $x \in K$

$$(T_\alpha^K(t)\varphi(x))^2 \leq \mathbb{E}[\varphi^2(X_\alpha(t, x)) \mathbb{1}_{\tau_x^K \geq t}] \leq \mathbb{E}[\tilde{\varphi}^2(X_\alpha(t, x)) \mathbb{1}_{\tau_x^K \geq t}] \leq T_\alpha(t)(\tilde{\varphi}^2)(x).$$

Since μ is invariant for $T_\alpha(t)$, it follows that

$$\begin{aligned} \int_K (T_\alpha^K(t)\varphi(x))^2 d\mu &\leq \int_K T_\alpha(t)(\tilde{\varphi}^2)(x) d\mu \\ &\leq \int_H T_\alpha(t)(\tilde{\varphi}^2) d\mu \leq \int_H T_\alpha(t)(\tilde{\varphi}^2) d\mu \leq \int_H \tilde{\varphi}^2 d\mu = \int_K \varphi(x)^2 d\mu. \end{aligned}$$

The conclusion follows. \square

We shall denote by L_α^K the infinitesimal generator of $T_\alpha^K(t)$ in $L^2(K, \mu)$.

Proposition 3.7. *For any $f \in L^2(K, \mu)$ and $t > 0$ we have*

$$\lim_{\varepsilon \rightarrow 0} (P_\alpha^\varepsilon(t)\tilde{f})|_K = T_\alpha^K(t)f, \quad \text{in } L^2(K, \mu) \quad (3.13)$$

and, for $\lambda > 0$,

$$\lim_{\varepsilon \rightarrow 0} (R(\lambda, M_\alpha^\varepsilon)\tilde{f})|_K = (\lambda - L_\alpha^K)^{-1}f, \quad \text{in } L^2(K, \mu). \quad (3.14)$$

Proof. Let $f \in C_b(H)$. By Proposition 3.5, $P_\alpha^\varepsilon f$ converges pointwise to $T_\alpha^K(t)f$ in K . Moreover, $|(P_\alpha^\varepsilon(t)f)(x)| \leq \|f\|_\infty$, $|(T_\alpha^K(t)f)(x)| \leq \|f\|_\infty$ for each $x \in K$ and $t > 0$. By dominated convergence, $\lim_{\varepsilon \rightarrow 0} \|P_\alpha^\varepsilon(t)f - T_\alpha^K(t)f\|_{L^2(K, \mu)} = 0$.

Let now $f \in L^2(K, \mu)$. Since $C_b(H)$ is dense in $L^2(H, \mu)$, there is a sequence $(f_n) \subset C_b(H)$ such that

$$\|\tilde{f} - f_n\|_{L^2(H, \mu)} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Then we have

$$\begin{aligned} \|T_\alpha^K(t)f - P_\alpha^\varepsilon(t)\tilde{f}\|_{L^2(K, \mu)} &\leq \|T_\alpha^K(t)(f - f_n)\|_{L^2(K, \mu)} \\ &+ \|T_\alpha^K(t)f_n - P_\alpha^\varepsilon(t)f_n\|_{L^2(K, \mu)} + \|P_\alpha^\varepsilon(t)(f_n - \tilde{f})\|_{L^2(K, \mu)} \\ &\leq \frac{2}{n} + \|T_\alpha^K(t)f_n - P_\alpha^\varepsilon(t)\tilde{f}_n\|_{L^2(K, \mu)}, \quad \forall n \in \mathbb{N}, \end{aligned}$$

and (3.13) follows.

To prove (3.14), we use the identity (in $L^2(H, \mu)$)

$$R(\lambda, M_\alpha^\varepsilon)\tilde{f} = \int_0^\infty e^{-\lambda t} P_\alpha^\varepsilon(t)\tilde{f} dt.$$

Taking the restrictions to K of both sides and using (3.13) we obtain

$$\lim_{\varepsilon \rightarrow 0} (R(\lambda, M_\alpha^\varepsilon)\tilde{f})|_K = \int_0^\infty e^{-\lambda t} T_\alpha^K(t)f dt,$$

which coincides with (3.14). \square

Theorem 3.8. *For every $\lambda > 0$ and $f \in L^2(K, \mu)$, the function $\varphi := R(\lambda, L_\alpha^K)f$ belongs to $\dot{W}_\alpha^{1,2}(K, \mu)$ and satisfies (2.10). Therefore, $T_\alpha^K(t)$ is the semigroup generated by M_α in $L^2(K, \mu)$.*

Proof. For $\varepsilon > 0$ define $\varphi_\varepsilon := R(\lambda, M_\alpha^\varepsilon)\tilde{f}$. By Proposition 3.3, φ_ε is the solution to (3.3), with f replaced by \tilde{f} . By Proposition 3.2, the $W_\alpha^{1,2}(H, \mu)$ -norm of φ_ε is bounded by a constant independent of ε . Therefore, there is a sequence $\varepsilon_k \rightarrow 0$ such that φ_{ε_k} converges weakly in $W_\alpha^{1,2}(H, \mu)$ to a function Φ . Let us prove that $\Phi = \tilde{\varphi}$.

For every $\psi \in L^2(K, \mu)$ we have

$$\int_K \Phi \psi d\mu = \lim_{k \rightarrow \infty} \int_H \varphi_{\varepsilon_k} \tilde{\psi} d\mu = \lim_{k \rightarrow \infty} \int_K \varphi_{\varepsilon_k} \psi d\mu = \int_K \varphi \psi d\mu$$

since, by Proposition 3.7, $\lim_{\varepsilon \rightarrow 0} \|\varphi_{\varepsilon}|_K - \varphi\|_{L^2(K, \mu)} = 0$. Then, $\Phi|_K = \varphi$.

Moreover,

$$\int_{K^c} \Phi^2 V d\mu = \int_H \Phi \cdot \Phi V \mathbb{1}_{K^c} d\mu = \lim_{k \rightarrow \infty} \int_H \varphi_{\varepsilon_k} \Phi V \mathbb{1}_{K^c} d\mu,$$

and by estimate (3.6) and the Hölder inequality we have

$$\left| \int_H \varphi_{\varepsilon_k} \Phi V \mathbb{1}_{K^c} d\mu \right| \leq \left(\int_{K^c} \varphi_{\varepsilon_k}^2 V d\mu \right)^{1/2} \left(\int_{K^c} \Phi^2 V d\mu \right)^{1/2} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It follows that $\Phi|_{K^c} = 0$. Therefore, $\Phi = \tilde{\varphi} \in W_{\alpha}^{1,2}(H, \mu)$, that is $\varphi \in \mathring{W}_{\alpha}^{1,2}(K, \mu)$.

For every $v \in \mathring{W}_{\alpha}^{1,2}(K, \mu)$ and $k \in \mathbb{N}$ we have (since $\int_H V \varphi_{\varepsilon_k} \tilde{v} d\mu = 0$)

$$\lambda \int_H \varphi_{\varepsilon_k} \tilde{v} d\mu + \frac{1}{2} \int_H \langle Q^{(1-\alpha)/2} D \varphi_{\varepsilon_k}, Q^{(1-\alpha)/2} D \tilde{v} \rangle d\mu = \int_H f v d\mu,$$

and letting $k \rightarrow \infty$ we obtain

$$\lambda \int_H \tilde{\varphi} \tilde{v} d\mu + \frac{1}{2} \int_H \langle Q^{(1-\alpha)/2} D \tilde{\varphi}, Q^{(1-\alpha)/2} D \tilde{v} \rangle d\mu = \int_H f \tilde{v} d\mu,$$

so that φ satisfies (2.10), and the statement follows. \square

3.3. Consequences. We list here some consequences of the results of this section, that hold for every $\alpha \in [0, 1]$.

- (i) $T_{\alpha}^K(t)$ is an analytic semigroup in $L^p(K, \mu)$ for every $p \in (1, \infty)$.
- (ii) The space $\mathring{W}_{\alpha}^{1,2}(K, \mu)$ coincides with the domain of $(I - L_{\alpha}^K)^{1/2}$.
- (iii) For each $f \in L^2(K, \mu)$ we have

$$\int_K |Q^{(1-\alpha)/2} D T_{\alpha}^K(t) f|^2 \mu(dx) \leq \frac{1}{\sqrt{t}} \int_K f^2(x) \mu(dx), \quad t > 0.$$

These statements follow in a standard way from the fact that the infinitesimal generator L_{α}^K of $T_{\alpha}^K(t)$ is the operator associated to the symmetric quadratic form \mathcal{Q}_{α} defined in (2.9), and that it is dissipative.

Less standard consequences are a Poincaré inequality in the space $\mathring{W}_{\alpha}^{1,2}(K, \mu)$ and the invertibility of L_{α}^K for $\alpha > 0$, proved in the next proposition.

Proposition 3.9. *For $\alpha \in (0, 1]$ the spaces $\mathring{W}_{\alpha}^{1,2}(K, \mu)$ and $D(L_{\alpha}^K)$ are compactly embedded in $L^2(K, \mu)$. Moreover $0 \in \rho(L_{\alpha}^K)$, and a Poincaré inequality holds in $\mathring{W}_{\alpha}^{1,2}(K, \mu)$,*

$$\|u\|_{L^2(K, \mu)} \leq C \int_K |Q^{(1-\alpha)/2} D u|^2 d\mu, \quad u \in \mathring{W}_{\alpha}^{1,2}(K, \mu).$$

Proof. Since the embedding $W_{\alpha}^{1,2}(H, \mu) \subset L^2(H, \mu)$ is compact by Proposition 2.2(b), the embedding $\mathring{W}_{\alpha}^{1,2}(K, \mu) \subset L^2(K, \mu)$ is compact too. Indeed, a sequence u_n is bounded in $\mathring{W}_{\alpha}^{1,2}(K, \mu)$ iff the sequence \tilde{u}_n is bounded in $W_{\alpha}^{1,2}(H, \mu)$. In this case, there is a subsequence of \tilde{u}_n that converges to a function $v \in L^2(H, \mu)$. Therefore, a subsequence of u_n converges to the restriction $v|_K$, in $L^2(K, \mu)$.

Since the domain $D(L_{\alpha}^K)$ is continuously embedded in $\mathring{W}_{\alpha}^{1,2}(K, \mu)$, it is compactly embedded in $L^2(K, \mu)$. Therefore, the spectrum of L_{α}^K consists of (nonpositive) eigenvalues. Let us prove that 0 is not an eigenvalue.

Let $u \in D(L_\alpha^K)$ be such that $L_\alpha^K u = 0$. Then

$$0 = \int_K u L_\alpha^K u \, d\mu = -\frac{1}{2} \int_K |Q^{(1-\alpha)/2} Du|^2 \, d\mu = -\frac{1}{2} \int_H |Q^{(1-\alpha)/2} D\tilde{u}|^2 \, d\mu,$$

and by the Poincaré inequality in $W_\alpha^{1,2}(H, \mu)$ (Proposition 2.2(a)) we have

$$\int_H (\tilde{u} - \int_H \tilde{u} \, d\mu)^2 \, d\mu = 0.$$

So, \tilde{u} is constant a.e. in H , but since it vanishes in K^c , whose measure is positive, then it vanishes a.e. in H . Therefore, $u = 0$.

This implies that the seminorm $u \mapsto (\int_K |Q^{(1-\alpha)/2} Du|^2 \, d\mu)^{1/2}$ is in fact an equivalent norm in $\dot{W}_\alpha^{1,2}(K, \mu)$, that is, a Poincaré inequality holds in $\dot{W}_\alpha^{1,2}(K, \mu)$. Indeed, since $-L_\alpha^K$ is invertible, also $(-L_\alpha^K)^{1/2}$ is invertible, so that the seminorm $u \mapsto \|(-L_\alpha^K)^{1/2} u\|_{L^2(K, \mu)} = \frac{1}{2} \int_K |Q^{(1-\alpha)/2} Du|^2 \, d\mu$ is an equivalent norm in $D((-L_\alpha^K)^{1/2}) = \dot{W}_\alpha^{1,2}(K, \mu)$; in other words there is $C > 0$ such that $\|u\|_{L^2(K, \mu)} \leq C \int_K |Q^{(1-\alpha)/2} Du|^2 \, d\mu$ for every $u \in \dot{W}_\alpha^{1,2}(K, \mu)$. \square

4. INTERIOR REGULARITY

In this section we prove an interior regularity result for the solution to (1.2) for $\alpha < 1$. We use the following lemma.

Lemma 4.1. *For every $\varphi \in D(L_\alpha)$ and for every $\beta \in \mathcal{E}_\alpha(H)$, the product $\varphi\beta$ belongs to the domain of L_α , and*

$$L_\alpha(\varphi\beta) = \beta L_\alpha \varphi + \varphi L_\alpha \beta + \langle Q^{1-\alpha} D\varphi, D\beta \rangle.$$

Proof. Since $\mathcal{E}_\alpha(H)$ is dense in $D(L_\alpha)$, there is a sequence $(\varphi_n) \subset \mathcal{E}_\alpha(H)$ that converges to φ in $D(L_\alpha)$. For every n , $\beta\varphi_n$ is still in $\mathcal{E}_\alpha(H)$, hence it belongs to $D(L_\alpha)$ and the statement follows easily. \square

Proposition 4.2. *Assume that*

$$\text{Tr } Q^{1-\alpha} = \sum_{k=1}^{\infty} \lambda_k^{1-\alpha} < \infty.$$

Then for every $y \in \mathring{K}$ and $r > 0$ such that $\text{dist}(B(y, r), \partial K) > 0$, the restriction to $B(y, r)$ of the solution φ to (1.2) belongs to $W_\alpha^{2,2}(B(y, r), \mu)$.

Proof. It is enough to prove that the statement holds for $y \in D(A^{\alpha/2})$. Indeed, since $D(A^{\alpha/2})$ is dense in H , for each $y \in \mathring{K}$ and $r > 0$ such that $\text{dist}(B(y, r), \partial K) > 0$ there are $y_1 \in \mathring{K} \cap D(A^{\alpha/2})$ and $r_1 > r$ such that $B(y, r) \subset B(y_1, r_1)$ and $\text{dist}(B(y_1, r), \partial K) > 0$.

So, let $y \in D(A^{\alpha/2})$ and let $r_1 > r$ be such that the ball $B(y, r_1)$ is contained in \mathring{K} . Let $\rho : \mathbb{R} \mapsto [0, 1]$ be a C^2 function such that

$$\rho(\xi) = 1, \quad \xi \leq r^2, \quad \rho(\xi) = 0, \quad \xi \geq r_1^2,$$

and define a cutoff function θ by

$$\theta(x) := \rho(|x - y|^2), \quad x \in H.$$

Our aim is to show that the product $\tilde{\varphi}\theta$ belongs to $W_\alpha^{2,2}(H, \mu)$. Since the restriction to $B(y, r)$ of $\tilde{\varphi}\theta$ coincides with the restriction to $B(y, r)$ of φ , the statement will follow.

The proof is in three steps. As a first step, we show that $\theta \in D(L_\alpha)$. Then we show that $\varphi_\varepsilon \theta$ belongs to $D(L_\alpha)$ for every $\varepsilon > 0$, where $\varphi_\varepsilon = R(\lambda, M_\alpha^\varepsilon)f$. Eventually, we prove that $\tilde{\varphi}\theta \in W_\alpha^{2,2}(H, \mu)$.

First step: $\theta \in D(L_\alpha)$. We approach each $x \in H$ by the sequence $x_n = \sum_{k=1}^n \langle x, e_k \rangle e_k$, and we consider the sequence of functions

$$\theta_n(x) := \rho(|x_n - y_n|^2), \quad x \in H, \quad n \in \mathbb{N}.$$

Each of them belongs to $D(L_\alpha)$. This is because it depends only on the first n coordinates, it is bounded and it has bounded first and second order derivatives, and in finite dimensions the inclusion $C_b^2(H) \subset D(L_\alpha)$ holds. Therefore, it is easy to see that there exists the limit $\lim_{t \rightarrow 0} (T_\alpha(t)\theta_n - \theta_n)/t = L_\alpha \theta_n$ in $L^2(H, \mu)$, where

$$\begin{aligned} L_\alpha \theta_n(x) &= \rho'(|x_n - y_n|^2) \left(\sum_{k=1}^n \lambda_k^{1-\alpha} - \sum_{k=1}^n \lambda_k^{-\alpha} \langle x, e_k \rangle \langle x - y, e_k \rangle \right) \\ &\quad + 2\rho''(|x_n - y_n|^2) \langle Q^{1-\alpha}(x_n - y_n), x_n - y_n \rangle. \end{aligned} \quad (4.1)$$

Letting $n \rightarrow \infty$, $\rho'(|x_n - y_n|^2)$ and $\rho''(|x_n - y_n|^2) \langle Q^{1-\alpha}(x_n - y_n), x_n - y_n \rangle$ converge in $L^2(H, \mu)$ to $\rho'(|x - y|^2)$ and to $\rho''(|x - y|^2) \langle Q^{1-\alpha}(x - y), x - y \rangle$, respectively, by dominated convergence. The sum $\sum_{k=1}^n \lambda_k^{-\alpha} \langle x, e_k \rangle \langle x - y, e_k \rangle$ converges too. Indeed, for $p < q \in \mathbb{N}$ we have

$$\begin{aligned} &\left\| \sum_{k=p}^q \lambda_k^{-\alpha} \langle x, e_k \rangle \langle x - y, e_k \rangle \right\|_{L^2(H, \mu)} \\ &\leq \sum_{k=p}^q \|\lambda_k^{-\alpha/2} \langle x, e_k \rangle\|_{L^2(H, \mu)} \|\lambda_k^{-\alpha/2} \langle x - y, e_k \rangle\|_{L^2(H, \mu)} \\ &= \sum_{k=p}^q \lambda_k^{(1-\alpha)/2} \lambda_k^{-\alpha/2} (\lambda_k + |\langle y, e_k \rangle|^2)^{1/2} \\ &\leq \sum_{k=p}^q \lambda_k^{(1-\alpha)/2} (\lambda_k^{(1-\alpha)/2} + \lambda_k^{-\alpha/2} |\langle y, e_k \rangle|) \\ &\leq \sum_{k=p}^q \lambda_k^{1-\alpha} + \frac{1}{2} (\lambda_k^{1-\alpha} + \lambda_k^{-\alpha} |\langle y, e_k \rangle|^2), \end{aligned}$$

where $\sum_{k=1}^\infty \lambda_k^{1-\alpha} < \infty$ by assumption, and $\sum_{k=1}^\infty \lambda_k^{-\alpha} |\langle y, e_k \rangle|^2 < \infty$ because $y \in D(A^{\alpha/2})$. Therefore,

$$\exists L^2(H, \mu) - \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_k^{-\alpha} \langle x, e_k \rangle \langle x - y, e_k \rangle := \langle x, A^\alpha(x - y) \rangle.$$

(Note that $\langle x, A^\alpha(x - y) \rangle$ is not defined pointwise). It follows that $\rho'(|x_n - y_n|^2) \cdot \sum_{k=1}^n \lambda_k^{-\alpha} \langle x, e_k \rangle \langle x - y, e_k \rangle$ converges to $\rho'(|x - y|^2) \langle x, A^\alpha(x - y) \rangle$ in $L^2(H, \mu)$. Since L_α is closed, $\theta \in D(L_\alpha)$.

Second step: $\varphi_\varepsilon \theta$ belongs to $D(L_\alpha)$.

Since $\varphi_\varepsilon \in D(L_\alpha)$ and $\mathcal{E}_\alpha(H)$ is a core of L_α , there is a sequence of exponential functions β_n that converges to φ_ε in $D(L_\alpha)$. Since θ is bounded, $\beta_n\theta$ converges to $\varphi_\varepsilon\theta$ in $L^2(H, \mu)$. By Lemma 4.1, $\beta_n\theta$ belongs to $D(L_\alpha)$ for every n , and we have

$$L_\alpha(\beta_n\theta) = \beta_n L_\alpha\theta + \theta L_\alpha\beta_n + \langle Q^{1-\alpha}D\beta_n, D\theta \rangle.$$

As $n \rightarrow \infty$, β_n converges to φ_ε , $L_\alpha\beta_n$ converges to $L_\alpha\varphi_\varepsilon$, and $\langle Q^{1-\alpha}D\beta_n, D\theta \rangle = \langle Q^{(1-\alpha)/2}D\beta_n, Q^{(1-\alpha)/2}D\theta \rangle$ converges to $\langle Q^{(1-\alpha)/2}D\varphi_\varepsilon, Q^{(1-\alpha)/2}D\theta \rangle$ in $L^2(H, \mu)$ since $D(L_\alpha) \subset W_\alpha^{1,2}(H, \mu)$ and $Q^{(1-\alpha)/2}D\theta$ is bounded. Therefore, $L_\alpha(\beta_n\theta)$ converges in $L^2(H, \mu)$, and since L_α is closed, $\varphi_\varepsilon\theta$ belongs to $D(L_\alpha)$ and

$$L_\alpha(\theta\varphi_\varepsilon) = (L_\alpha\theta)\varphi_\varepsilon + \langle Q^{(1-\alpha)/2}D\theta, Q^{(1-\alpha)/2}D\varphi_\varepsilon \rangle + \theta L_\alpha\varphi_\varepsilon. \quad (4.2)$$

Third step: $\tilde{\varphi}\theta$ belongs to $W_\alpha^{2,2}(H, \mu)$. Using (4.2) and (3.3) we get

$$\lambda\theta\varphi_\varepsilon - L_\alpha(\theta\varphi_\varepsilon) = \theta\tilde{f} - (L_\alpha\theta)\varphi_\varepsilon - \langle Q^{(1-\alpha)/2}D\theta, Q^{(1-\alpha)/2}D\varphi_\varepsilon \rangle := f_{1,\varepsilon}.$$

The L^2 norm of the right hand side $f_{1,\varepsilon}$ is bounded by a constant independent of ε . Therefore, $\|\theta\varphi_\varepsilon\|_{D(L_\alpha)}$ is bounded by a constant independent of ε , and since $D(L_\alpha)$ is continuously embedded in $W_\alpha^{2,2}(H, \mu)$, also $\|\theta\varphi_\varepsilon\|_{W_\alpha^{2,2}(H, \mu)}$ is .

Let $\{\varepsilon_k\}$ be the sequence used in the proof of Proposition 3.8, so that φ_{ε_k} converges weakly in $W_\alpha^{1,2}(H, \mu)$ to $\tilde{\varphi}$. Possibly taking a further subsequence, $(\theta\varphi_{\varepsilon_k})$ converges weakly in $W_\alpha^{2,2}(H, \mu)$ to a function u that belongs to $W_\alpha^{2,2}(H, \mu)$. Then $u = \theta\tilde{\varphi}$; indeed, for each $\psi \in L^2(H, \mu)$ we have

$$\int_H u \psi d\mu = \lim_{k \rightarrow \infty} \int_H \theta\varphi_{\varepsilon_k} \psi d\mu = \lim_{k \rightarrow \infty} \int_H \theta\tilde{\varphi} \psi d\mu.$$

So, $\theta\tilde{\varphi} \in W_\alpha^{2,2}(H, \mu)$. □

5. DOMAINS WITH SMOOTH BOUNDARIES

In this section we assume that

$$K = \{x \in H : g(x) \leq 1\}$$

where $g : H \mapsto \mathbb{R}$ is a C^1 function that belongs to $D(L_0)$ and satisfies (A.8). Moreover we assume that $\sup g > 1$, so that K is a proper subset of H , and $\inf g < 1$, so that the interior part of K is not empty and the surface measure $d\sigma$ is well defined in the boundary Σ of K , $\Sigma = \{x \in H : g(x) = 1\}$. See the Appendix, to which we refer for the definition and properties of surface measures.

The aim of this section is to give a reasonable definition of the trace at ∂K of any function in $W_\alpha^{1,2}(H, \mu)$, and to show that the functions in $\mathring{W}_\alpha^{1,2}(H, \mu)$ have null trace at ∂K . This implies that $R(\lambda, L_\alpha^K)f$ satisfies the Dirichlet boundary condition in (1.2) in the sense of the trace for every $f \in L^2(K, \mu)$, and that $T_\alpha^K(t)f$ has null trace at the boundary for every $t > 0$ and $f \in L^2(K, \mu)$.

As a first step we prove integration formulas for functions in the core $\mathcal{E}_0(H)$.

Proposition 5.1. *Let $k \in \mathbb{N}$ be such that $D_k g / |Q^{1/2}Dg| \in W_0^{2,2}(H, \mu)$. Then for every $\varphi \in \mathcal{E}_0(H)$ we have*

$$\int_K D_k \varphi d\mu = \frac{1}{\lambda_k} \int_K x_k \varphi d\mu + \int_\Sigma \frac{D_k g}{|Q^{1/2}Dg|} \varphi d\sigma. \quad (5.1)$$

If $|Q^{1/2}Dg| \in W_0^{2,2}(H, \mu)$, then for every $\varphi \in \mathcal{E}_0(H)$ we have

$$\int_{\Sigma} \varphi^2 |Q^{1/2} Dg| d\sigma_1 \begin{cases} = \int_K \varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle d\mu + \int_K L_0 g \varphi^2 d\mu & (a) \\ = - \int_{K^c} \varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle d\mu - \int_{K^c} L_0 g \varphi^2 d\mu & (b) \end{cases} \quad (5.2)$$

Proof. For small $\varepsilon > 0$ define the pathwise linear function θ_ε by

$$\theta_\varepsilon(\xi) := \begin{cases} 2, & \xi \leq 1 - \varepsilon, \\ \frac{1}{\varepsilon}(1 - \xi) + 1, & 1 - \varepsilon < \xi < 1 + \varepsilon, \\ 0, & \xi \geq 1 + \varepsilon. \end{cases},$$

and set

$$\rho_\varepsilon(x) := \theta_\varepsilon(g(x)), \quad x \in H.$$

Since θ_ε is Lipschitz continuous, then $\rho_\varepsilon \in W_0^{1,2}(H, \mu)$ ([2, Rem. 5.2.1]). Then the product $\rho_\varepsilon \varphi$ belongs to $W_0^{1,2}(H, \mu)$ and $D_k(\rho_\varepsilon \varphi) = \theta'_\varepsilon(g(x)) D_k g(x) \varphi(x) + \rho_\varepsilon(x) D_k \varphi(x)$, so that

$$\int_H (D_k \varphi) \rho_\varepsilon d\mu - \frac{1}{\varepsilon} \int_{1-\varepsilon < g < 1+\varepsilon} \varphi D_k g d\mu = \frac{1}{\lambda_k} \int_H x_k \varphi \rho_\varepsilon d\mu, \quad k \in \mathbb{N}. \quad (5.3)$$

Let us prove (5.1). Letting $\varepsilon \rightarrow 0$, ρ_ε converges pointwise to $2\mathbb{1}_K$ in $H \setminus \Sigma$, whose measure is 1. Since $\rho_\varepsilon \leq 2$, by dominated convergence we get

$$\exists \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{1-\varepsilon < g < 1+\varepsilon} \varphi D_k g d\mu = \int_K D_k \varphi d\mu - \frac{1}{\lambda_k} \int_K x_k \varphi d\mu.$$

Let us identify the limit in the left hand side as a boundary integral. Since $\varphi D_k g |Q^{1/2} Dg|^{-1} \in W_0^{2,2}(H, \mu)$, by Remark A.7 we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{1-\varepsilon < g < 1+\varepsilon} \varphi D_k g d\mu = \int_{\Sigma} \frac{D_k g}{|Q^{1/2} Dg|} \varphi d\sigma$$

and (5.1) follows.

Let us prove (5.2)(a). For every $\varepsilon > 0$ and $k \in \mathbb{N}$, the function $\rho_\varepsilon \varphi^2 D_k g$ still belongs to $W_0^{1,2}(H, \mu)$. Therefore we may replace φ in (5.3) by $\lambda_k \varphi^2 D_k g$, and summing over k (recall Lemma 2.1), we obtain

$$\begin{aligned} & \int_H 2\varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle \rho_\varepsilon d\mu + \int_H 2L_0 g \varphi^2 \rho_\varepsilon d\mu \\ &= \frac{1}{\varepsilon} \int_{1-\varepsilon < g < 1+\varepsilon} \varphi^2 |Q^{1/2} Dg|^2 d\mu \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ as before, by dominated convergence we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_H \varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle \rho_\varepsilon d\mu &= \int_K \varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle d\mu, \\ \lim_{\varepsilon \rightarrow 0} \int_H L_0 g \varphi^2 \rho_\varepsilon d\mu &= \int_K L_0 g \varphi^2 d\mu. \end{aligned}$$

Therefore, there exists the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{1-\varepsilon < g < 1+\varepsilon} \varphi^2 |Q^{1/2} Dg|^2 d\mu = \int_K \varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle d\mu + \int_K L_0 g \varphi^2 d\mu$$

that we identify as a boundary integral. Indeed, since $\varphi^2|Q^{1/2}Dg| \in W_0^{2,2}(H, \mu)$, by Remark A.7 we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{1-\varepsilon < g < 1+\varepsilon} \varphi^2|Q^{1/2}Dg|^2 d\mu = \int_{\Sigma} \varphi^2|Q^{1/2}Dg| d\sigma.$$

So, (5.2)(a) holds. To prove (5.2)(b), we may follow the same procedure replacing K by K^c and θ_ε by

$$\tilde{\theta}_\varepsilon(\xi) := \begin{cases} 0, & \xi \leq 1 - \varepsilon, \\ \frac{1}{\varepsilon}(\xi - 1) + 1, & 1 - \varepsilon < \xi < 1 + \varepsilon, \\ 2, & \xi \geq 1 + \varepsilon, \end{cases}$$

or else, we may use the equality

$$\begin{aligned} & \int_{K^c} \varphi \langle Q^{1/2}D\varphi, Q^{1/2}Dg \rangle d\mu + \int_{K^c} L_0g \varphi^2 d\mu \\ &= - \int_K \varphi \langle Q^{1/2}D\varphi, Q^{1/2}Dg \rangle d\mu - \int_K L_0g \varphi^2 d\mu \end{aligned}$$

that follows from

$$\int_H L_0g \varphi^2 d\mu = -\frac{1}{2} \int_H \langle Q^{1/2}Dg, Q^{1/2}D(\varphi^2) \rangle d\mu = - \int_H \langle Q^{1/2}Dg, Q^{1/2}D\varphi \rangle \varphi d\mu$$

(see formula (2.6)). \square

As a second step, with the aid of Proposition 5.1 we prove an integration by parts formula in $W_0^{1,2}(H, \mu)$ and we define the *trace* $\varphi|_\Sigma$ at the boundary Σ of any function in $W_0^{1,2}(H, \mu)$.

Corollary 5.2. *Assume that $|Q^{1/2}Dg| \in W_0^{2,2}(H, \mu)$, and that $|Q^{1/2}Dg|$ is bounded and L_0g has at most linear growth either on K or on K^c . Then for every $\varphi \in W_0^{1,2}(H, \mu)$ there exists $\psi \in L^2(\Sigma, \sigma)$ with the following property: for each sequence $(\varphi_n) \in \mathcal{E}_0(H)$ such that $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_{W_0^{1,2}(H, \mu)} = 0$, the sequence $(\varphi_n|Q^{1/2}Dg|_\Sigma^{1/2})$ converges to ψ in $L^2(\Sigma, \sigma)$.*

Proof. It is sufficient to recall formula (5.2) and Lemma 2.1. \square

Note that the assumptions of Corollary 5.2 are satisfied by the functions g in Example A of the Appendix.

Definition 5.3. Under the assumptions of Corollary 5.2, for each $\varphi \in W_0^{1,2}(H, \mu)$ the trace of φ at Σ is defined by

$$\varphi|_\Sigma = \frac{\psi}{|Q^{1/2}Dg|^{1/2}},$$

where ψ is given by Corollary 5.2.

Note that in general $\varphi|_\Sigma$ does not belong to $L^2(\Sigma, \sigma)$, because $|Q^{1/2}Dg|^{-1/2}$ may be unbounded in Σ . Of course, if $|Q^{1/2}Dg|^{-1/2}$ is bounded in Σ (that is, if $\inf_\Sigma |Q^{1/2}Dg| > 0$), then $\varphi|_\Sigma \in L^2(\Sigma, \sigma)$ for every $\varphi \in W_0^{1,2}(H, \mu)$ and the mapping $W_0^{1,2}(H, \mu) \mapsto L^2(\Sigma, \sigma)$, $\varphi \mapsto \varphi|_\Sigma$ is continuous.

In general, we have the following lemma.

Lemma 5.4. *Under the assumptions of Corollary 5.2, for every $\varphi \in W_0^{1,2}(H, \mu)$, $\varphi|_\Sigma \in L^1(\Sigma, \sigma)$ and the mapping $W_0^{1,2}(H, \mu) \mapsto L^1(\Sigma, \sigma)$, $\varphi \mapsto \varphi|_\Sigma$ is continuous.*

Proof. Since $\varphi|_{\Sigma} = \psi|Q^{1/2}Dg|^{-1/2}$ with $\psi \in L^2(\Sigma, \sigma)$, it is sufficient to prove that $|Q^{1/2}Dg|^{-1/2} \in L^2(\Sigma, \sigma)$. The assumptions $\|Q^{1/2}D^2gQ^{1/2}\|_{\mathcal{L}(H)}/|Q^{1/2}Dg|^2 \in L^2(H, \mu)$ and $|Q^{1/2}Dg|^{-1} \in L^4(H, \mu)$, that are contained in assumption (A.8), imply that the function $\tilde{\varphi} := |Q^{1/2}Dg|^{-1}$ belongs to $W_0^{1,2}(H, \mu)$. By Corollary 5.2, $\tilde{\varphi}|Q^{1/2}Dg|^{1/2} = |Q^{1/2}Dg|^{-1/2}$ has trace in $L^2(\Sigma, \sigma)$. \square

Corollary 5.5. *Let the assumptions of Corollary 5.2 be satisfied. The following statements hold for every $\alpha \in [0, 1]$.*

- (i) *If $D_k g/|Q^{1/2}Dg| \in W_0^{2,2}(H, \mu)$, for every $\varphi \in W_{\alpha}^{1,2}(H, \mu)$ the integration by parts formula (5.1) holds.*
- (ii) *If $\varphi \in \dot{W}_{\alpha}^{1,2}(K, \mu)$, its trace at Σ_1 vanishes.*

Proof. Since $W_{\alpha}^{1,2}(H, \mu) \subset W_0^{1,2}(H, \mu)$, and $\dot{W}_{\alpha}^{1,2}(K, \mu) \subset \dot{W}_0^{1,2}(K, \mu)$, it is enough to prove that the statements hold for $\alpha = 0$.

- (i) It is sufficient to approach every $\varphi \in W_0^{1,2}(H, \mu)$ by a sequence $(\varphi_n) \subset \mathcal{E}_0(H)$, and to recall Lemma 5.4.
- (ii) If $\varphi \in \dot{W}_0^{1,2}(K, \mu)$, it vanishes a.e. in K^c , and formula (5.2)(b) yields the statement. \square

APPENDIX A. SURFACE INTEGRALS

We consider level surfaces of smooth functions g . We refer to [2, §6.10], where the functions g under consideration belong to the space $W^{\infty}(H, \mu)$ defined by

$$W^{\infty}(H, \mu) := \bigcap_{k \in \mathbb{N}, p > 1} W^{k,p}(H, \mu)$$

and $W^{k,p}(H, \mu)$ is the completion of the smooth cylindrical functions⁽¹⁾ in the norm

$$\|f\|_{k,p} := \|f\|_{L^p(H, \mu)} + \sum_{j=1}^k \left(\int_H \left[\sum_{i_1, \dots, i_j \geq 1} (\lambda_{i_1} \cdots \lambda_{i_k} D_{i_1} \cdots D_{i_k} f(x))^2 \right]^{p/2} \mu(dx) \right)^{1/p}$$

(In particular, the spaces $W^{k,2}(H, \mu)$ coincide with our $W_0^{k,2}(H, \mu)$ for $k = 1, 2$).

Another assumption is

$$|Q^{1/2}Dg|^{-1} \in \bigcap_{p > 1} L^p(H, \mu).$$

Our aim here is to give a simplified presentation of surface measures in the case of a Hilbert space setting, under less heavy (although less elegant) assumptions on g .

For any continuous $g : H \mapsto \mathbb{R}$ and r in the range of g let us define the level sets

$$\Sigma_r := \{x \in H : g(x) = r\}.$$

We shall define probability measures on the surfaces Σ_r with r in the interior part of the range of g . To this aim, a first step is the study of the image of μ on \mathbb{R} under the mapping g , defined by

$$(\mu \circ g^{-1})(I) := \mu(g^{-1}(I)), \quad I \in \mathcal{B}(\mathbb{R}).$$

¹that is, functions of the type $f(x) = \varphi(\langle x, x_1 \rangle, \dots, \langle x, x_n \rangle)$ with $x_1, \dots, x_n \in H$ and $\varphi \in C_b^{\infty}(\mathbb{R}^n)$.

We shall give sufficient conditions for $\mu \circ g^{-1}$ have continuous (in fact, $W^{1,2}$) density k with respect to the Lebesgue measure. Similarly, for $\rho \in L^1(H, \mu)$ we shall consider the signed measure

$$(\rho\mu)(B) := \int_B \rho(x)\mu(dx), \quad B \in \mathcal{B}(H)$$

and its image under the mapping g ,

$$(\rho\mu \circ g^{-1})(I) := (\rho\mu)(g^{-1}(I)), \quad I \in \mathcal{B}(\mathbb{R}),$$

and we shall give sufficient conditions for $\rho\mu \circ g^{-1}$ have continuous density k_ρ with respect to the Lebesgue measure. A key role will be played by the function ψ defined by

$$\psi := \frac{L_0 g}{|Q^{1/2} Dg|^2} - \frac{\langle Q^{1/2} D^2 g Q^{1/2} \cdot Q^{1/2} Dg, Q^{1/2} Dg \rangle}{|Q^{1/2} Dg|^4}, \quad (\text{A.1})$$

if $g \in D(L_0)$. We shall use the following lemma.

Proposition A.1. *Let $g \in D(L_0)$ be such that $|Q^{1/2} Dg|^{-1} \in L^4(H, \mu)$. Then*

- (a) $\mu \circ g^{-1}$ is absolutely continuous with respect to the Lebesgue measure.
- (b) If a function $\rho \in W_0^{1,1}(H, \mu)$ is such that

$$\psi\rho \in L^1(H, \mu), \quad \frac{|Q^{1/2} D\rho|}{|Q^{1/2} Dg|} \in L^1(H, \mu), \quad (\text{A.2})$$

where ψ is defined in (A.1), then $\rho\mu \circ g^{-1}$ is absolutely continuous with respect to the Lebesgue measure.

Proof. To prove statement (a) we shall show that there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}} \varphi'(r)(\mu \circ g^{-1})(dr) \right| \leq C \|\varphi\|_\infty, \quad \varphi \in C_b^1(\mathbb{R}). \quad (\text{A.3})$$

For each $k \in \mathbb{N}$ we have

$$D_k(\varphi \circ g)(x) = \varphi'(g(x)) D_k g(x), \quad x \in H, \quad (\text{A.4})$$

so that

$$\langle D(\varphi \circ g)(x), Q Dg(x) \rangle = (\varphi' \circ g)(x) |Q^{1/2} Dg(x)|^2, \quad x \in H, \quad (\text{A.5})$$

i.e.

$$(\varphi' \circ g)(x) = \frac{\langle Q^{1/2} D(\varphi \circ g)(x), Q^{1/2} Dg(x) \rangle}{|Q^{1/2} Dg(x)|^2}, \quad \text{a.e. } x \in H. \quad (\text{A.6})$$

Therefore,

$$\int_{\mathbb{R}} \varphi'(r)(\mu \circ g^{-1})(dr) = \int_H \varphi' \circ g d\mu = \int_H \frac{\sum_k \lambda_k D_k(\varphi \circ g)(x) D_k g(x)}{|Q^{1/2} Dg(x)|^2} d\mu.$$

Integrating by parts and recalling that

$$D_k \left(\frac{1}{|Q^{1/2} Dg|^2} \right) = -2 \frac{\sum_i \lambda_i D_i g D_{ik} g}{|Q^{1/2} Dg|^4} \quad (\text{A.7})$$

we obtain

$$\begin{aligned}
\int_H \varphi' \circ g \, d\mu &= - \int_H \varphi \circ g \sum_k \lambda_k D_k \left(\frac{D_k g}{|Q^{1/2} Dg|^2} \right) d\mu + \int_H \varphi \circ g \sum_k \frac{x_k D_k g}{|Q^{1/2} Dg|^2} d\mu \\
&= - \int_H \varphi \circ g \sum_k \lambda_k \left(\frac{D_{kk} g}{|Q^{1/2} Dg|^2} - 2 D_k g \frac{\sum_i \lambda_i D_i g D_{ik} g}{|Q^{1/2} Dg|^4} \right) d\mu \\
&\quad + \int_H \varphi \circ g \sum_k \frac{x_k D_k g}{|Q^{1/2} Dg|^2} d\mu \\
&= -2 \int_H (\varphi \circ g)(x) \psi(x) d\mu,
\end{aligned}$$

where the function ψ is defined in (A.1). The first addendum in ψ , $L_0 g / |Q^{1/2} Dg|^2$, belongs to $L^1(H, \mu)$ since both $L_0 g$ and $1/|Q^{1/2} Dg|^2$ are in $L^2(H, \mu)$. Concerning the second addendum we have

$$\frac{|\langle Q^{1/2} D^2 g Q^{1/2} \cdot Q^{1/2} Dg, Q^{1/2} Dg \rangle|}{|Q^{1/2} Dg|^4} \leq \frac{\|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)}}{|Q^{1/2} Dg|^2}.$$

Recalling that there exists $C_0 > 0$ such that ([2, Thm. 5.7.1])

$$\|x \mapsto \|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)}\|_{L^2(H, \mu)} \leq C_0 \|g\|_{D(L_0)},$$

it follows that the second addendum in ψ belongs to $L^1(H, \mu)$. Then formula (A.3) follows, with $C = \|\psi\|_{L^1(H, \mu)} \leq \text{const.}$ ($\|g\|_{D(L_0)} + \| |Q^{1/2} Dg|^{-1} \|_{L^4(H, \mu)}$).

We prove statement (b) by the same procedure, replacing μ by $\rho\mu$. For every $\varphi \in C_b^1(\mathbb{R})$ we have

$$\begin{aligned}
\int_H (\varphi' \circ g) \rho \, d\mu &= \int_H \sum_k \lambda_k D_k (\varphi \circ g)(x) D_k g(x) \frac{\rho(x)}{|Q^{1/2} Dg(x)|^2} d\mu \\
&= \int_H \varphi \circ g \left(-2\psi\rho - \frac{\langle Q^{1/2} Dg, Q^{1/2} D\rho \rangle}{|Q^{1/2} Dg(x)|^2} \right) d\mu
\end{aligned}$$

where ψ is the function defined in (A.1). Assumption (A.2) implies that the functions $\psi\rho$ and $\langle Q^{1/2} Dg, Q^{1/2} D\rho \rangle / |Q^{1/2} Dg(x)|^2$ belong to $L^1(H, \mu)$. Then,

$$\left| \int_{\mathbb{R}} \varphi'(r) (\mu \circ g^{-1})(dr) \right| = \left| \int_H (\varphi' \circ g) \rho \, d\mu \right| \leq C \|\varphi\|_{\infty}, \quad \varphi \in C_b^1(\mathbb{R})$$

with $C = 2\|\psi\rho\|_{L^1} + \left\| \frac{|Q^{1/2} D\rho|}{|Q^{1/2} Dg|} \right\|_{L^1}$. The statement follows. \square

Proposition A.2. *Let the assumptions of Proposition A.1 hold. Then:*

- (a) *If the function ψ defined in (A.1) belongs to $W_0^{1,2}(H, \mu)$, then the density k of $\mu \circ g^{-1}$ belongs to $W^{1,1}(\mathbb{R})$.*
- (b) *If $\rho \in W_0^{1,1}(H, \mu)$ satisfies (A.2) and moreover, setting*

$$\rho_1 := 2\psi\rho + \frac{\langle Q^{1/2} Dg, Q^{1/2} D\rho \rangle}{|Q^{1/2} Dg|^2}$$

we have $\rho_1 \in W_0^{1,1}(H, \mu)$, $\psi \rho_1 \in L^1(H, \mu)$, $\frac{|Q^{1/2} D \rho_1|}{|Q^{1/2} D g|} \in L^1(H, \mu)$, then $k_\rho \in W^{1,1}(\mathbb{R})$.

Proof. To prove statement (a) we shall show that there is $C_1 > 0$ such that

$$\left| \int_{\mathbb{R}} \varphi''(r)(\mu \circ g^{-1})(dr) \right| \leq C_1 \|\varphi\|_\infty, \quad \varphi \in C_b^2(\mathbb{R}).$$

Indeed, this implies that k is weakly differentiable with $k' \in L^1(\mathbb{R})$.

Differentiating (A.4) we get

$$D_{kk}(\varphi \circ g)(x) = \varphi''(g(x))(D_k g(x))^2 + \varphi'(g(x))D_{kk}g(x), \quad x \in H,$$

and summing over k

$$\text{Tr}(QD^2(g \circ \varphi)) = \varphi''(g(x))|Q^{1/2}Dg(x)|^2 + \varphi'(g(x))\text{Tr}(QD^2g(x))$$

so that

$$\begin{aligned} \varphi'' \circ g &= \frac{\text{Tr}(QD^2(\varphi \circ g))}{|Q^{1/2}Dg|^2} - (\varphi' \circ g) \frac{\text{Tr}(QD^2g)}{|Q^{1/2}Dg|^2} \\ &= \frac{2L_0(\varphi \circ g) + \langle x, D(\varphi \circ g) \rangle}{|Q^{1/2}Dg|^2} - (\varphi' \circ g) \frac{2L_0g + \langle x, Dg \rangle}{|Q^{1/2}Dg|^2} \\ &= \frac{2L_0(\varphi \circ g)}{|Q^{1/2}Dg|^2} - (\varphi' \circ g) \frac{2L_0g}{|Q^{1/2}Dg|^2}. \end{aligned}$$

Using again (A.7) we get

$$\begin{aligned} \int_H (\varphi'' \circ g) d\mu &= \int_H \left(- \langle Q^{1/2}D(\varphi \circ g), Q^{1/2}D(|Q^{1/2}Dg|^{-2}) \rangle - 2(\varphi' \circ g) \frac{L_0g}{|Q^{1/2}Dg|^2} \right) d\mu \\ &= \int_H \varphi' \circ g \left(\langle Q^{1/2}Dg, 2 \frac{Q^{1/2}D^2g Q^{1/2} \cdot Q^{1/2}Dg}{|Q^{1/2}Dg|^4} \rangle - 2 \frac{L_0g}{|Q^{1/2}Dg|^2} \right) d\mu \\ &= -2 \int_H (\varphi' \circ g) \psi d\mu, \end{aligned}$$

where ψ is defined in (A.1). Then we may use Proposition A.1, with $\rho = \psi$. By assumption, $\psi \in W_0^{1,2}(H, \mu) \subset W_0^{1,1}(H, \mu)$, moreover $\psi^2 \in L^1(H, \mu)$ and $\frac{|Q^{1/2}D\psi|}{|Q^{1/2}Dg|} \in L^1(H, \mu)$ since $|Q^{1/2}D\psi| \in L^2(H, \mu)$, $|Q^{1/2}Dg|^{-1} \in L^2(H, \mu)$. We get $|\int_H (\varphi' \circ g) \psi d\mu| \leq C \|\psi\|_{W_0^{1,2}(H, \mu)} \|\varphi\|_\infty$, and statement (a) follows.

Concerning statement (b), the proof is similar, replacing μ by $\rho\mu$. For every $\varphi \in C_b^2(\mathbb{R})$ we have

$$\begin{aligned} \int_H (\varphi'' \circ g) \rho d\mu &= \int_H \left(\frac{2\rho L_0(\varphi \circ g)}{|Q^{1/2}Dg|^2} - 2(\varphi' \circ g) \frac{\rho L_0g}{|Q^{1/2}Dg|^2} \right) d\mu \\ &= \int_H \left(- \langle Q^{1/2}D(\varphi \circ g), Q^{1/2}D\left(\frac{\rho}{|Q^{1/2}Dg|^2}\right) \rangle - 2(\varphi' \circ g) \frac{\rho L_0g}{|Q^{1/2}Dg|^2} \right) d\mu \end{aligned}$$

$$\begin{aligned}
&= \int_H \varphi' \circ g \left(\langle Q^{1/2} Dg, 2 \frac{Q^{1/2} D^2 g Q^{1/2} \cdot Q^{1/2} Dg}{|Q^{1/2} Dg|^4} \rangle - 2 \frac{L_0 g}{|Q^{1/2} Dg|^2} \right) \rho d\mu \\
&\quad - \int_H \varphi' \circ g \frac{\langle Q^{1/2} Dg, Q^{1/2} D\rho \rangle}{|Q^{1/2} Dg|^2} d\mu \\
&= - \int_H \varphi' \circ g \left(2\psi \rho + \frac{\langle Q^{1/2} Dg, Q^{1/2} D\rho \rangle}{|Q^{1/2} Dg|^2} \right) d\mu = - \int_H \varphi' \circ g \rho_1 d\mu,
\end{aligned}$$

where the function ρ_1 satisfies the assumptions of Proposition A.1(b). We obtain $|\int_H (\varphi' \circ g) \rho_1 d\mu| \leq C \|\varphi\|_\infty$ and the statement follows. \square

One can play with ρ and g in order that the assumptions of Proposition A.2(b) are satisfied. In the next proposition we give sufficient conditions that are useful for the sequel.

Proposition A.3. *The assumptions of Proposition A.2(b) are satisfied by every $\rho \in W_0^{2,2}(H, \mu)$ provided $g \in D(L_0)$ is such that*

$$\begin{cases} |Q^{1/2} Dg|^{-1} \in L^4(H, \mu), & \psi \in W_0^{1,4}(H, \mu), \\ \frac{\|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)}}{|Q^{1/2} Dg|^2} \in L^2(H, \mu), & \frac{\|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)}}{|Q^{1/2} Dg|^3} \in L^2(H, \mu). \end{cases} \quad (\text{A.8})$$

In this case there exists $C_2 > 0$, depending only on g , such that

$$\left| \int_{\mathbb{R}} \varphi''(r) (\rho \mu \circ g^{-1})(dr) \right| \leq C_2 \|\rho\|_{W_0^{2,2}(H, \mu)} \|\varphi\|_\infty, \quad \varphi \in C_b^2(\mathbb{R}).$$

Consequently, if $\rho_n \rightarrow \rho$ in $W_0^{2,2}(H, \mu)$ then $k_{\rho_n} \rightarrow k_\rho$ in $W^{1,1}(\mathbb{R})$, hence $k_{\rho_n} \rightarrow k_\rho$ in $L^\infty(\mathbb{R})$.

Proof. Since $\psi \in L^2$ and $|Q^{1/2} Dg|^{-1} \in L^4$, then $\rho_1 \in L^1$. Computing $Q^{1/2} D\rho_1$ we obtain

$$\begin{aligned}
Q^{1/2} D\rho_1 &= \\
&= \rho Q^{1/2} D\psi + \psi Q^{1/2} D\rho - \frac{Q^{1/2} D^2 g Q^{1/2} \cdot Q^{1/2} D\rho + Q^{1/2} D^2 \rho Q^{1/2} \cdot Q^{1/2} Dg}{|Q^{1/2} Dg|^2} \\
&\quad + 2 \langle Q^{1/2} Dg, Q^{1/2} D\rho \rangle \frac{Q^{1/2} D^2 g Q^{1/2} \cdot Q^{1/2} Dg}{|Q^{1/2} Dg|^4}.
\end{aligned}$$

Estimating each addendum we get

- $\rho |Q^{1/2} D\psi| \in L^1$, since $|Q^{1/2} D\psi| \in L^2$;
- $\psi |Q^{1/2} D\rho| \in L^1$, since $\psi \in L^2$;
- $\frac{\|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)} |Q^{1/2} D\rho|}{|Q^{1/2} Dg|^2} \in L^1$, since $\frac{\|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)}}{|Q^{1/2} Dg|^2} \in L^2$;
- $\frac{\|Q^{1/2} D^2 \rho Q^{1/2}\|_{\mathcal{L}(H)}}{|Q^{1/2} Dg|} \in L^1$, since $\frac{1}{|Q^{1/2} Dg|} \in L^2$;
- $|Q^{1/2} D\rho| \frac{\|Q^{1/2} D^2 g Q^{1/2}\|_{\mathcal{L}(H)}}{|Q^{1/2} Dg|^2} \in L^1$, as above.

Therefore $\rho_1 \in L^1$, and $\|\rho_1\|_{L^1(H,\mu)} \leq c\|\rho\|_{W_0^{2,2}(H,\mu)}$.

The assumptions $\psi \in L^4$, $\frac{1}{|Q^{1/2}Dg|} \in L^4$ imply that $\psi\rho_1 \in L^1$.

To check that $\frac{|Q^{1/2}D\rho_1|}{|Q^{1/2}Dg|} \in L^1$ we redo the estimates above, dividing each term by $|Q^{1/2}Dg|$. We get

- $\rho \frac{|Q^{1/2}D\psi|}{|Q^{1/2}Dg|} \in L^1$, since $|Q^{1/2}D\psi| \in L^4$ and $\frac{1}{|Q^{1/2}Dg|} \in L^4$;
- $\psi \frac{|Q^{1/2}D\rho|}{|Q^{1/2}Dg|} \in L^1$, since $\psi \in L^4$ and $\frac{1}{|Q^{1/2}Dg|} \in L^4$;
- $\frac{\|Q^{1/2}D^2g\|_{\mathcal{L}(H)}|Q^{1/2}D\rho|}{|Q^{1/2}Dg|^3} \in L^1$, since $\frac{\|Q^{1/2}D^2g\|_{\mathcal{L}(H)}}{|Q^{1/2}Dg|^3} \in L^2$;
- $\frac{\|Q^{1/2}D^2\rho\|_{\mathcal{L}(H)}}{|Q^{1/2}Dg|^2} \in L^1$, since $\frac{1}{|Q^{1/2}Dg|} \in L^4$;
- $|Q^{1/2}D\rho| \frac{\|Q^{1/2}D^2g\|_{\mathcal{L}(H)}}{|Q^{1/2}Dg|^3} \in L^1$, as above.

Therefore, the norms $\|\psi\rho_1\|_{L^1}$ and $\|\frac{|Q^{1/2}D\rho_1|}{|Q^{1/2}Dg|}\|_{L^1}$ are bounded by $c\|\rho\|_{W_0^{2,2}(H,\mu)}$. Applying Proposition A.2(b) the statement follows. \square

Example. Let us consider some simple examples.

- (a) $g(x) = \langle b, x \rangle$, with $|b| = 1$,
- (b) $g(x) = \langle Tx, x \rangle$, with $T \in \mathcal{L}(H)$, $Te_k = t_k e_k$ for each $k \in \mathbb{N}$ and $t_k \neq 0$ for infinitely many k ,
- (c) $g(x) = \sum_{k=1}^{13} x_k^2$.

In all these cases g satisfies the conditions of Proposition A.3.

Proof. In case (a) we have $Dg = b$, $D^2g = 0$ so that $L_0g = -\langle b, x \rangle/2 = -g/2$ and

$$\psi = -\frac{\langle b, x \rangle}{2|Q^{1/2}b|^2}$$

which belongs to $W_0^{1,4}(H,\mu)$. The other conditions of Proposition A.3 are obviously satisfied.

In case (b) we have $Dg(x) = 2Tx$, $D^2g(x) = 2T$ so that $L_0g = \text{Tr}[QT] - g$ and

$$\psi(x) = \frac{\text{Tr}[QT] - \langle Tx, x \rangle}{2|Q^{1/2}Tx|^2} - \frac{\langle Q^2T^3x, x \rangle}{|Q^{1/2}Tx|^2}. \quad (\text{A.9})$$

Since $t_k \neq 0$ for infinitely many k , then $x \mapsto |Q^{1/2}Dg(x)|^{-1}$ belongs to all spaces $L^p(H,\mu)$. Indeed, $|Q^{1/2}Dg(x)|^2 \geq 4 \sum_{k=1}^N \lambda_i t_k^2 x_k^2$ where N is so large that at least $[p] + 1$ addenda do not vanish. The other assumptions of Remark A.3 are easily seen to be satisfied.

In case (c) we still have $g(x) = \langle Tx, x \rangle$ with $T \in \mathcal{L}(H)$, $Tx = \sum_{k=1}^{13} x_k e_k$, so that $t_k \neq 0$ only for $k = 1, \dots, 13$. However, $|Q^{1/2}Dg(x)|^{-1} \leq c_0(\sum_{k=1}^{13} x_k^2)^{-1/2}$ with $c_0 = 1/\min\{\lambda_k^{1/2} : k = 1, \dots, 13\}$ so that $|Q^{1/2}Dg|^{-1} \in L^p(H,\mu)$ for every $p < 13$. The function ψ is still given by (A.9) on $\text{span}\{e_1, \dots, e_{13}\}$ and it belongs to $L^p(H,\mu)$ for every $p < 13/3$, in particular it belongs to $L^4(H,\mu)$, as well as $|Q^{1/2}D\psi|^{-1}$. The other conditions of Proposition A.3 are easily seen to be satisfied. \square

In cases (a) and (b) with $T = I$ it is possible to give a representation formula for k that shows that $k \in C^\infty$, see [10]. In case (c) we have $|Q^{1/2}Dg(x)|^{-1} \geq c_1(\sum_{k=1}^{13} x_k^2)^{-1/2}$ with $c_1 = 1/\max\{\lambda_k^{1/2} : k = 1, \dots, 13\}$ so that $|Q^{1/2}Dg|^{-1} \notin L^p(H, \mu)$ for $p \geq 13$.

The construction of the surface measures goes as follows. First, one constructs surface measures depending explicitly on g by an approximation procedure.

One fixes once and for all a convex compact set K which is symmetric with respect to the origin and has positive measure, say $\mu(K) > 1/2$. Such a K does exist. Indeed, it is well known that there are compact sets \tilde{K} with positive (arbitrarily close to 1) measure (a simple proof is e.g. in [3, Thm. 6.2]). The absolute convex hull K of \tilde{K} is compact, symmetric with respect to the origin and contains \tilde{K} , so that $\mu(K) \geq \mu(\tilde{K})$.

Then we need a regular cutoff function. The proof of its existence follows closely [2, Prop. 5.4.12], with a few simplifications due to our Hilbert space setting.

Lemma A.4. *Let $K \subset H$ be compact, convex, symmetric with respect to the origin, with $\mu(K) > 1/2$. Then there exists a function $\theta \in W^\infty(H, \mu)$ such that $\theta \equiv 1$ on K , $\theta = 0$ a.e. outside $2K$ and $0 \leq \theta(x) \leq 1$ for all $x \in H$.*

Proof. By the 0 – 1 law (e.g., [2, Thm. 2.5.5]), the vector space E spanned by K has measure 1. Consequently, $\lim_{m \rightarrow \infty} \mu(mK) = 1$. Fix $m \in \mathbb{N}$ such that

$$\mu(mK) > \frac{8}{9}.$$

Let us consider the Minkowski functional defined by $p_K(x) := \inf\{\alpha > 0 : x \in \alpha K\}$ for $x \in E$, and the function $d(x) := \inf\{p_K(x - y) : y \in K\}$ if $x \in E$, $d(x) = 1$ if $x \notin E$. We modify it setting

$$\varphi(x) = 1 - h(d(x)), \quad x \in H,$$

where $h(t) = t$ for $t \leq 1$ and $h(t) = 1$ for $t \geq 1$. The function φ is Borel measurable, has values between 0 and 1, $\varphi \equiv 1$ on K and $\varphi \equiv 0$ outside E and outside $2K$. We regularize it applying $T_0(t)$, where $t > 0$ is chosen such that

$$1 - e^{-t/2} < \frac{1}{8}, \quad m\sqrt{1 - e^{-t}} < \frac{1}{8}.$$

Since $\varphi \in \mathcal{B}_b(H)$, then $T_0(t)\varphi \in W^\infty(H, \mu)$ (e.g., [2, Prop. 5.4.8]).

Moreover,

$$T_0(t)\varphi(x) \geq \frac{2}{3} \quad \forall x \in K, \quad T_0(t)\varphi(x) \leq \frac{3}{5} \quad \forall x \in E \setminus 2K. \quad (\text{A.10})$$

Indeed, let $x \in K$. Then $e^{-t/2}x \in K$, and for each $y \in mK$ we have $\sqrt{1 - e^{-t}}y \in K/8$. The sum $e^{-t/2}x + \sqrt{1 - e^{-t}}y$ belongs to $9K/8$, so that $d(e^{-t/2}x + \sqrt{1 - e^{-t}}y) \leq 1/8$ and therefore $\varphi(e^{-t/2}x + \sqrt{1 - e^{-t}}y) \geq 7/8$. Since $\mu(H \setminus mK) \leq 1/9$, we get $T_0(t)\varphi(x) \geq 7/8 - 1/9 > 2/3$. Let now $x \in E \setminus 2K$. Since $e^{-t/2} > 7/8$, $e^{-t/2}x \notin 7K/4$ and consequently for every $y \in mK$ the sum $e^{-t/2}x + \sqrt{1 - e^{-t}}y$ does not belong to $7K/4 - K/8 = 13K/8$. Therefore, $d(e^{-t/2}x + \sqrt{1 - e^{-t}}y) \geq 5/8$, so that $\varphi(e^{-t/2}x + \sqrt{1 - e^{-t}}y) \leq 3/8$. Again since $\mu(H \setminus mK) \leq 1/9$, we get $T_0(t)\varphi(x) \leq 3/8 + 1/9 = 35/72 < 3/5$, and (A.10) is proved.

Now fix a function $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta(t) = 0$ for $t \leq 3/5$, $\eta(t) = 1$ for $t \geq 2/3$, and set

$$\theta(x) = \eta(T_0(t)\varphi(x)), \quad x \in H.$$

The function θ is what we were looking for. It has values between 0 and 1, it belongs to $W^\infty(H, \mu)$, $\theta(x) = 1$ for $x \in K$ and $\theta(x) = 0$ for $x \in E \setminus 2K$. Since $\mu(E) = 1$, then $\theta(x) = 0$ for almost all $x \in H \setminus 2K$. The statement follows. \square

Now we fix $\varphi_0 \in C_c^\infty(\mathbb{R})$ with $0 \leq \varphi_0 \leq 1$, $\int_{\mathbb{R}} \varphi_0(t) dt = 1$ and $\varphi_0 \equiv 1$ in a neighborhood of 0, $\varphi_0 \equiv 0$ outside $(-1, 1)$. Then for each $r \in \mathbb{R}$ the sequence $\{\varphi_0(j(t-r))dt/j\}$ converges weakly to the Dirac measure δ_r .

For each r in the interior part of $g(H)$ we set

$$\theta_n(x) = \theta\left(\frac{x}{n}\right), \quad x \in H; \quad \varphi_j(t) = \frac{\varphi_0(j(t-r))}{j}, \quad j \in \mathbb{N}, t \in \mathbb{R}.$$

The following proposition is proved in [2]. Since in the Hilbert space case there are not simplifications with respect to the general setting of [2], we refer to [2, Lemma 6.10.1, Thm. 6.10.2] for the proof.

Proposition A.5. (a) *For each $n \in \mathbb{N}$, the sequence of measures*

$$\nu_{n,j}(dx) = \theta_n(x) \frac{\varphi_j(g(x))}{k(g(x))} \mu(dx)$$

converges weakly to a measure ν_n concentrated on $\Sigma_r := g^{-1}(r)$. Moreover, for each continuous $f \in W_0^{2,2}(H)$ we have

$$\int_H f d\nu_n = \int_{\Sigma_r} f d\nu_n = \frac{k_f \theta_n(r)}{k(r)}. \quad (\text{A.11})$$

(b) *In its turn, the sequence ν_n converges weakly to a probability measure $\sigma_r^{(g)}$ concentrated on Σ_r , such that for each continuous $f \in W_0^{2,2}(H)$ we have*

$$\int_H f d\sigma_r^{(g)} = \int_{\Sigma_r} f d\sigma_r^{(g)} = \frac{k_f(r)}{k(r)}. \quad (\text{A.12})$$

Definition A.6. For every Borel bounded function $\varphi : H \mapsto \mathbb{R}$ and for every r in the interior part of $g(H)$ we set

$$\int_{\Sigma_r} \varphi d\sigma_r := k(r) \int_{\Sigma_r} \varphi |Q^{1/2} Dg| d\sigma_r^{(g)}.$$

Remark A.7. It is easy to see that for every $f : H \mapsto \mathbb{R}$ such that $f|Q^{1/2} Dg| \in W_0^{2,2}(H, \mu) \cap C(H)$ we have

$$\int_{\Sigma_r} f d\sigma_r = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{r-\varepsilon \leq g(x) \leq r+\varepsilon} f |Q^{1/2} Dg| d\mu.$$

Indeed, applying Proposition A.2 we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{r-\varepsilon \leq g(x) \leq r+\varepsilon} f |Q^{1/2} Dg| d\mu &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} d(f|Q^{1/2} Dg| \circ \mu) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} k_{f|Q^{1/2} Dg|}(t) dt = k_{f|Q^{1/2} Dg|}(r). \end{aligned}$$

On the other hand, by Proposition A.5(b) we have

$$k_{f|Q^{1/2} Dg|}(r) = k(r) \int_{\Sigma_r} f |Q^{1/2} Dg| d\sigma_r^{(g)}$$

and the right hand side is just $\int_{\Sigma_r} f d\sigma_r$ by definition.

REFERENCES

- [1] V. Barbu, G. Da Prato, L. Tubaro, *Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space*, Ann. Probab. **37** (2009), 1427–1458.
- [2] V.I. Bogachev, *Gaussian Measures*, American Mathematical Society, Providence, 1998.
- [3] G. Da Prato, *An introduction to Infinite-Dimensional Analysis*, Springer-Verlag, Berlin, 2006.
- [4] G. Da Prato, B. Goldys, J. Zabczyk, *Ornstein-Uhlenbeck semigroups in open sets of Hilbert spaces*, C. R. Acad. Sci. Paris, **325**, 433–438, 1997.
- [5] G. Da Prato, J. Zabczyk, *Second Order Partial Differential Equations in Hilbert Spaces*, London Mathematical Society, Lecture Notes, **293**, Cambridge University Press, Cambridge, 2002.
- [6] J. D. Deuschel, D. Stroock, *Large Deviations*, Academic Press, San Diego, 1984.
- [7] E. B. Dynkin, *Markov Processes*, Vol I, Springer-Verlag, Berlin, 1965.
- [8] A. Friedman, *Stochastic differential equations and applications. Vol.1*, Academic Press, New York, 1975.
- [9] L. Gross, *Potential Theory in Hilbert spaces* J. Funct. Analysis **1**, (1965), 123–189.
- [10] A. Hertle, *Gaussian surface measures and the Radon transform on separable Banach spaces*, Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), 513–531, Lecture Notes in Math. **794**, Springer-Verlag, Berlin, 1980.
- [11] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.
- [12] P. Kotelenetz, *A submartingale type inequality with applications to stochastic evolution equations*, Stochastics, **8** (1982), 139–51.
- [13] A. Lunardi, G. Metafune, D. Pallara, *Dirichlet boundary conditions for elliptic operators with unbounded drift*, Proc. Amer. Math. Soc. **133** (2005), 2625–2635.
- [14] P. Malliavin, *Stochastic analysis*, Springer-Verlag, Berlin, 1997.
- [15] A. Talarczyk, *Dirichlet problem for parabolic equations on Hilbert spaces*, Studia Math. **141** (2000), 109–142.
- [16] L. Tubaro, *An estimate of Burkholder type for stochastic processes defined by the stochastic integral*, Stochastic Anal. Appl., **2** (1984), 187–192.

SCUOLA NORMALE SUPERIORE, PIAZZA DEI CAVALIERI, 7, 56126 PISA, ITALY
E-mail address: `g.daprato@sns.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PARMA, VIALE G.P. USBERTI, 53/A, 43124 PARMA, ITALY
E-mail address: `alessandra.lunardi@unipr.it`